DESCRIPTIVE GEOMETRY

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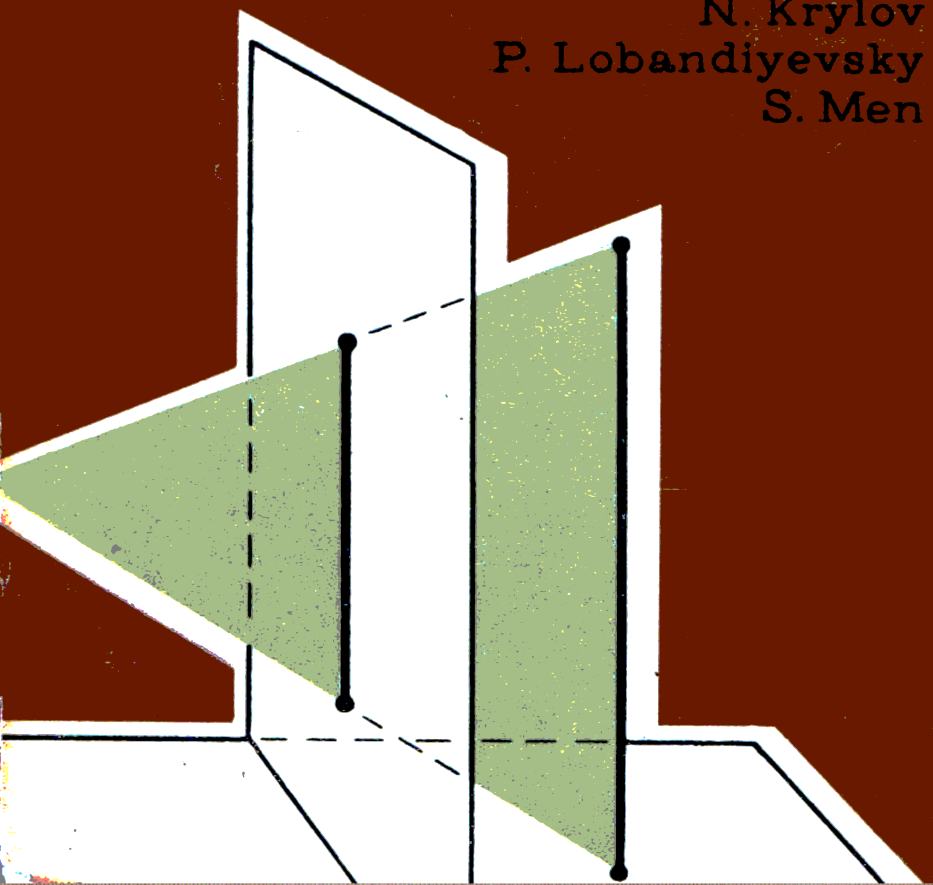

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INTRODUCTION

Descriptive geometry is a branch of geometry in which the spatial forms of objects are represented on a plane.

The basic objectives of descriptive geometry are:

(1) the study of various methods of representing three-dimensional figures on a plane;

(2) the investigation of the geometric properties of figures and solids on the basis of given images.

Since the time Gaspard Monge (1746-1818) placed the subject on a firm mathematical basis, descriptive geometry has occupied a definite place in higher education as a science needed both by engineers and architects.

Its importance as an applied discipline lies in the fact that it helps one to understand and read the language of engineering drawings.

The study of descriptive geometry is a great aid in developing three-dimensional visualization and habits of correct logical reasoning. By developing the ability to visualize three-dimensional objects on the basis of one-dimensional drawings, descriptive geometry helps the future engineer in the study of specialized subjects and trains him in the creative work of designing.

Sec. 1. The Method of Projections

The operation called projection underlies the construction of any representation. It consists in the following. An arbitrary point S in space is chosen as the so-called centre of projection (Fig. 1) and a plane K, which does not pass through S, is taken as the projection plane (picture plane). To project point A on plane K, draw the line-segment SA through the centre of projection S and produce it to its intersection with the plane K at point A_K. The point A_K is called the central projection of the point A, the line segment SA, a projector. The projections of all the points of a figure form the projection of the figure.

Projectors drawn through all the points of a curvilinear figure form a projecting conic surface. Thus, the projection of a curvilinear figure is the line of intersection of the projecting surface and the picture plane.

It will be noted that when projecting the line-segment AB, the projecting surface will be a plane. The latter will intersect the plane K along the line-segment A_KB_K, which will be the projection of the line-segment AB.

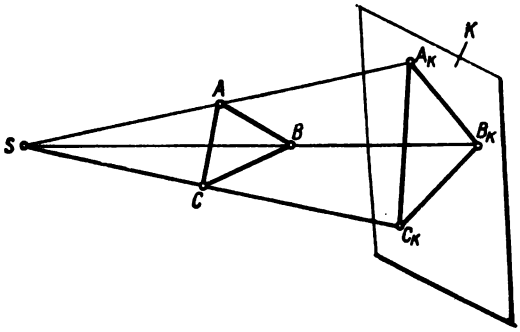


Fig. 1

Thus, a straight line not passing through the point S is projected in the form of a straight line.

To construct the projection of a figure it is not always necessary to project all the points of the figure. For instance, to project the triangle ABC, it is sufficient to construct projections A_K, B_K, C_K (three

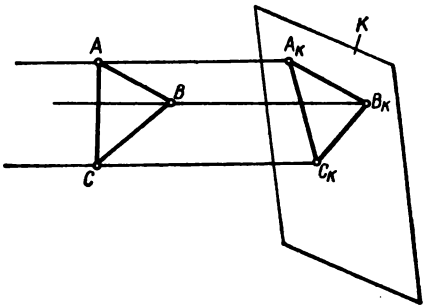


Fig. 2

points). In constructing the projection of an n-gon or any polyhedron, it is sufficient to determine the projections of the vertices.

In practical work, we frequently place the centre of projection at infinity. In this case, the projectors are parallels (Fig. 2), and the projections of points, figures and solids are called parallel projections. These, in turn, are subdivided into oblique and rectangular projec-

tions. In the former case, the direction of projection forms an angle with the projection plane not equal to 90°; in the latter case, the angle is a right angle.

Sec. 2. Projection Drawings

The representation of an object obtained by means of central or parallel projection is called a projection drawing. This drawing should comply with the following requirements.

1. A projection drawing should be pictorial, which means that it should help to visualize the object in three dimensions.
2. Each drawing should exactly (uniquely) define the form and position of the object being depicted. This property of the drawing is called "reversibility".
3. The image of the object should be convenient for reading the dimensions.
4. The construction of the image should be simple.

Referring to Figs. 1 and 2, it is evident that these drawings (central and parallel projection) are not "reversible". Separately, the central projection or parallel projection of a point does not define its position in space. Indeed, from the given projection A_K it is not possible to define the position of point A for the simple reason that to each point A_K on the plane K there will correspond any point of the projector SA.

Later on, as we examine in detail the various techniques of representing images, we shall see how this indeterminacy is eliminated and how a one-to-one relationship is established between points in space and points on the plane K.

Sec. 3. Certain Properties of Parallel Projection

Let us establish which properties of a configuration of points, lines and plane figures are retained by their parallel projections. In Fig. 3, let the direction of projection be given by the arrow MN.

1. A line-segment not parallel to MN will be projected into a straight line, since all the rays that project points A, B, C (which lie on a single straight line) into points A_K, B_K, C_K, are situated in a single plane.
2. The ratio of the line-segments of AB is equal to the ratio of their projections: $\frac{AC}{CB} = \frac{A_KC_K}{C_KB_K}$, since AA_K is parallel to BB_K and BB_K is parallel to CC_K.
3. The projections of two parallel straight lines are parallel. Let AB be parallel to EF. If we assume that their projections A_KB_K and E_KF_K are not parallel, then their point of intersection on the plane K will correspond to the point of intersection of AB and EF, which cannot be, since AB and EF are parallel. Hence, A_KB_K is parallel to E_KF_K.

4. The ratio of line-segments of two parallel straight lines is equal to the ratio of their projections.

To illustrate, let AB be parallel to EF. Connecting A and E and drawing FD parallel to AE, we have a parallelogram $AEFD$. By virtue of Item 3, the quadrangle $A_KE_KF_KD_K$ is also a parallelogram, and therefore $E_KF_K=A_KD_K$.

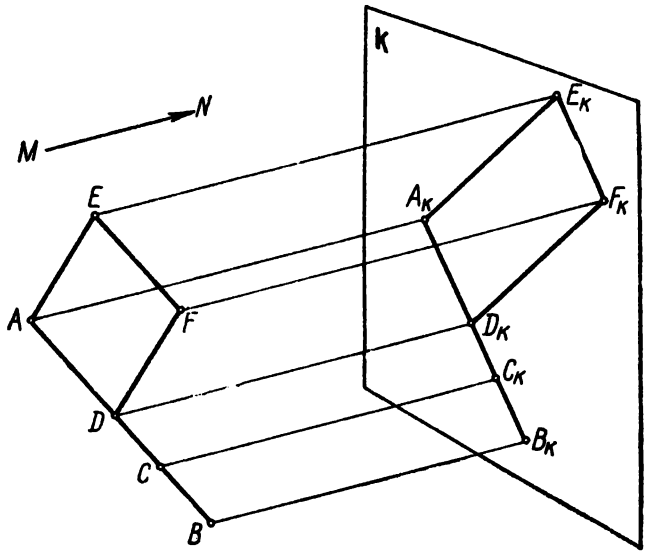


Fig. 3

But $\frac{AB}{AD}=\frac{A_KB_K}{A_KD_K}$, and since $AD=EF$ and $A_KD_K=E_KF_K$, we have $\frac{AB}{EF}=\frac{A_KB_K}{E_KF_K}$.

We shall have to come back to these properties of parallel projection many times during the consideration of various problems of our course.

Part one

ORTHOGONAL PROJECTIONS

The method of orthogonal projection consists in projecting an object on two mutually perpendicular planes by rays orthogonal (perpendicular) to these planes.

All the propositions proved in Sec. 3 will hold for orthogonal projections, which are a particular case of parallel projections.

*

CHAPTER I

PROJECTIONS OF A POINT

Sec. 4. The Orthogonal System of Two Planes of Projection. Orthographic Representation of a Point

One of the planes of an orthogonal projection is horizontal and is called the horizontal plane of projection H, the other is vertical V (Fig. 4). The planes H and V are infinite and opaque. The line of intersection of the planes of the projections is called the coordinate axis, designated Ox, the x-axis. The projection planes divide the space into four dihedral angles or quadrants, which are usually numbered as shown in Fig. 4.

When considering orthogonal projections, it is assumed that the viewer is located in the first quadrant at an infinite distance from the planes of projection. Since these planes are opaque, the viewer will see only those points, lines and figures situated within the confines of the first quadrant.

When constructing projections, bear in mind that the orthogonal projection of a point on a plane is the foot of the perpendicular dropped from the point onto the plane.

Fig. 4 shows a certain point A and its orthogonal projections a and a'.

The point a is called the horizontal projection of A, and the point a' (a prime) is its vertical projection. Each of them is the foot of the perpendicular from the point A to the respective planes H and V.

Let us now agree to denote points in space by capital letters A, B, C, etc., or Roman numerals I, II, III, etc., and their projections by the corresponding lower-case letters or Arabic numerals:

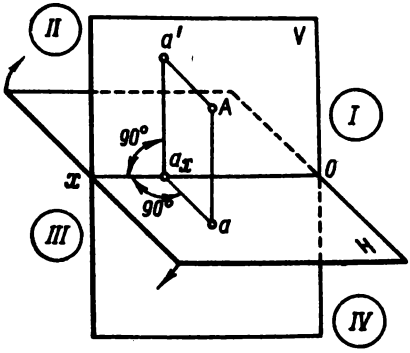


Fig. 4

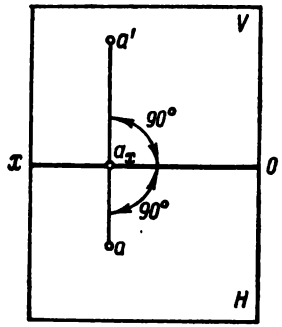


Fig. 5

a, b, c, ..., 1, 2, 3, ... for horizontal projections, and a', b', c', ..., 1′, 2′, 3′, ... for vertical projections.

It may be proven that *the projections of a point are always situated on straight lines perpendicular to the axis and intersecting this axis in one and the same point.* Indeed, the projecting rays Aa and Aa' define a plane perpendicular to the projection planes and to the line of their intersection, the x-axis. This plane intersects H and V along the line-segments aa_x and $a'a_x$, which form right angles with the x-axis and with each other, the angles having the vertex at the point a_x.

The converse is also true: if on planes of projection we have points a and a' located on line-segments intersecting the x-axis in a single point at right angles, they are then the projections of a certain point A. This point will be defined by the intersection of perpendiculars erected from a and a' to the planes H and V.

Note that the space position of the planes of projection may be different. For instance, both planes may be vertical, though mutually perpendicular. However, even in this case the above-proven proposition about the orientation of different projections of a point relative to the axis still holds.

To obtain a plane drawing consisting of the above-indicated projections, planes H and V are rotated to coincidence about the x-axis,

as shown by the arrows in Fig. 4. As a result, the front half-plane of H will coincide with the lower half-plane of V, and the rear half-plane of H with the upper half-plane of V.

A projection drawing on which the projection planes together with everything they contain are brought to coincidence in a specific manner is called orthographic representation. Fig 5 shows point A represented orthographically.

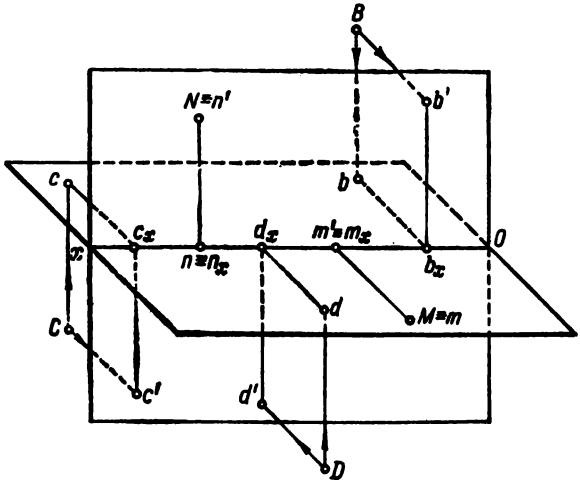


Fig. 6

In this method of bringing planes H and V into coincidence, the projections a and a' will be located on a single perpendicular to the x-axis, and the distance $a_x a$—from the horizontal projection of A to the x-axis—will be equal to the distance from the point A itself to the V plane, while the distance $a_x a'$—from the vertical projection of the point A to the x-axis—to the distance from the point A itself to the H plane.

Let us agree to call the straight lines connecting different projections of a point *lines of recall.*

The position of projections of points on the diagram depends on the quadrant in which the given point lies.

Thus, if point B lies in the second quadrant (Fig. 6), then after coincidence of the planes, both projections will be above the x-axis (Fig. 7).

If a point C lies in the third quadrant, its horizontal projection after coincidence of planes is above the axis, and its vertical projection

is below the x-axis. Finally, if point D is situated in the fourth quadrant, then both its projections will be below the x-axis. Figs. 6 and 7 show points M and N lying on projection planes.* In this position, the point coincides with one of its projections, while the other projection is found to lie on the x-axis. This peculiarity is reflected in

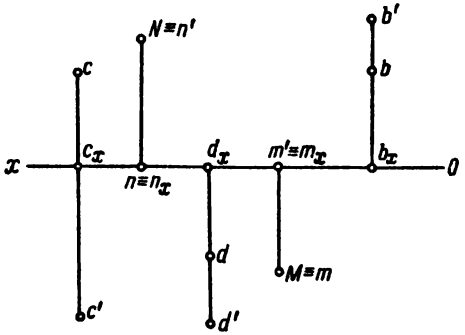


Fig. 7

the designation: a capital letter is given near the projection with which the point itself coincides.

Note also the case when both projections of a point coincide. This will occur if the point lies in the second or fourth quadrant at the same distance from the planes of projection. Both projections merge with the point itself if the latter is located on the x-axis.

Sec. 5. An Orthogonal System of Three Planes of Projection

It has been shown that two projections of a point define the position of the latter in space. Since every figure or solid is a set of points, it may be asserted that two orthogonal projections of an object (with suitable literal designations) fully define its shape.

However, when representing structural elements, machines and various engineering structures, it becomes necessary to build other projections in addition to the two basic projections. The sole purpose is to make the projection drawing more pictorial and readily visualizable.

A model of three projection planes is shown in Fig. 8. The third plane, perpendicular to H and V, is denoted by the letter W and is called the profile plane.

* In Figs. 6 and 7 and subsequent figures, coinciding points are indicated by an identity sign.

The projections of points on this plane are called profile projections. We shall designate them with double-prime lower-case letters or Arabic numerals (a'', b'', c'', ..., $1''$, $2''$, $3''$, ...).

The projection planes intersect in pairs and define three axes: x-, y-, z-axis, which may be regarded as a system of rectangular Cartesian coordinates in space with the origin at O. Each coordinate axis is divided into two half-lines by the point O. The system of signs given in Fig. 8 corresponds to the "right-handed system" of coordinates.

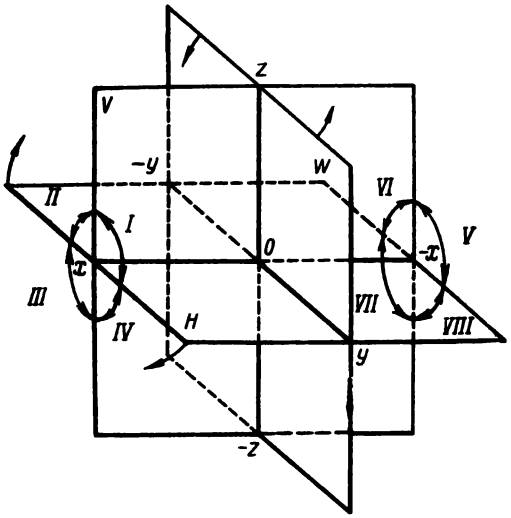


Fig. 8

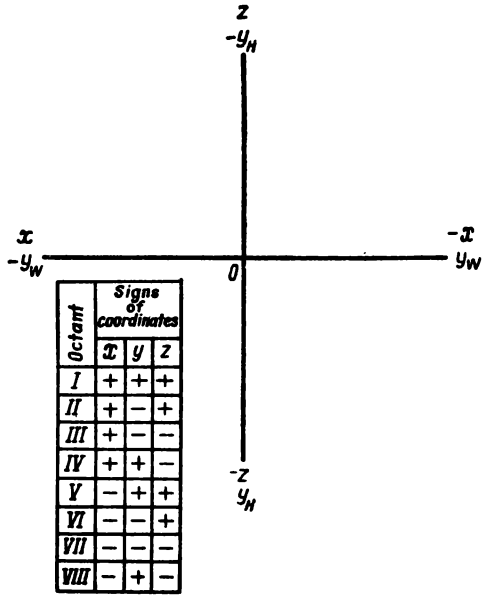
Three planes of projection divide space into eight trihedral angles, called octants. The octants are numbered as in Fig. 8.

As before, we consider that the viewer regarding the object is located in the first octant.

To obtain a drawing, planes H and W are rotated, as indicated in Fig. 8, to coincidence with plane V. As a result of rotation, the front half-plane of H is found to merge with the lower half-plane of V, and the rear half-plane of H coincides with the upper half-plane of V. In a 90° rotation about the z-axis, the front half-plane of W will coincide with the right half-plane of V, and the rear half-plane of W with the left half-plane of V.

Fig. 9 shows the final form in which all projection planes are coincident. In this figure, the x- and z-axes, which lie in the stationary plane V, are depicted only once, while the y-axis is indicated twice.

This is due to the fact that as the y-axis rotates together with the plane H, it coincides (orthographically) with the z-axis, but the same axis coincides with the x-axis when rotated together with the plane W.



Octant	Signs of coordinates		
	x	y	z
I	+	+	+
II	+	−	+
III	+	−	−
IV	+	+	−
V	−	+	+
VI	−	−	+
VII	−	−	−
VIII	−	+	−

Fig. 9

In Fig. 9, this peculiarity is indicated by the subscripts H and W on the y-axis.

Henceforward, negative semiaxes ($-x$, $-y$, $-z$) will not be indicated on orthographic drawings.

Sec. 6. Three Coordinates and Three Projections of a Point

Coordinates are numbers that correspond to a point, thus defining the position of the point in space or in a plane.

In three-dimensional space, the position of a point is defined by means of the rectangular Cartesian coordinates x, y and z.

The coordinate x is the *abscissa*, y the *ordinate* and z the *z-coordinate*. The abscissa x defines the distance of a given point to the plane W, the ordinate y to the plane V, and the z-coordinate to the plane H. Taking Fig. 8 as the reference system for defining a point, let us compile a table (see Fig. 9) of the signs of the coordinates in all eight oc-

tants. A point A in space defined by its coordinates is written thus: $A\ (x, y, z)$.

If $x=5$, $y=4$ and $z=6$, then we write $A\ (5, 4, 6)$. This point A, all the coordinates of which are positive, lies in the first octant.

To construct a point and its projections, let us make use of a spatial model (Fig. 10) in the form of a coordinate parallelepiped, which in our case will always be rectangular. First, lay off on the coordinate axes segments from O respectively equal to 5, 4, and 6 units of length. Using these segments (Oa_x, Oa_y, Oa_z) as edges, construct a rectangular parallelepiped. The vertex opposite the origin will represent

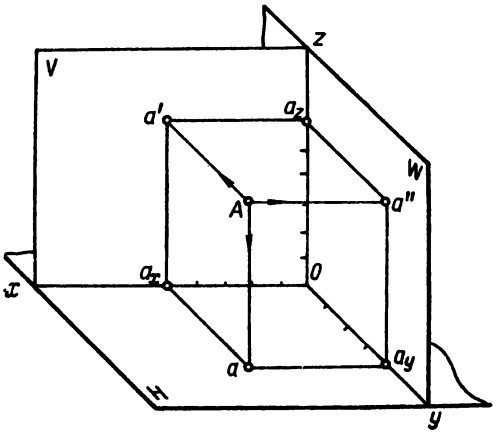


Fig. 10

the given point A. It will readily be seen that to determine point A, it is sufficient to construct only three edges of the parallelepiped, for example, Oa_x, a_xa and aA or Oa_y, a_ya and aA, etc. These edges form a coordinate polygonal line, the length of each segment of which is defined by the appropriate coordinate of the point.

However, this parallelepiped also enables us to construct all three orthogonal projections of the point A as well.

The three edges of the parallelepiped, which intersect at A, are rays projecting the point onto the planes H, V and W.

Each of the orthogonal projections of A is defined by only two coordinates, since it is located in a plane.

Thus, the horizontal projection a is defined by the coordinates x and y, the vertical projection a' by the coordinates x and z, the profile projection a'', by the coordinates y and z. But any two projections are defined by three coordinates. That is why the specification of

a point by two projections is equivalent to the specification of the point by three coordinates.

In an orthographic representation (Fig. 11) where all projection planes are coincident, the projections a and a' will be on a single perpendicular to the x-axis, and the projections a' and a'' will be situated on a single perpendicular to the z-axis.

As for the projections a and a'', we see that they are joined by the line-segments aa_y and $a''a_y$ which are perpendicular to the y-axis. But since this axis occupies two positions in the orthographic representation, the line-segment aa_y cannot be an extension of $a''a_y$.

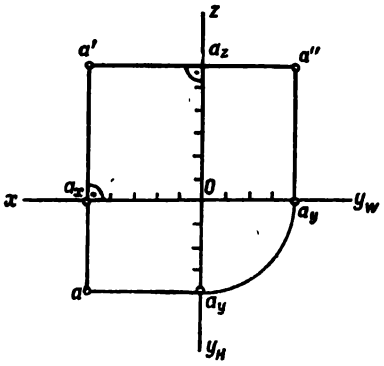


Fig. 11

The construction of the projections of a point A (5, 4, 6) on a drawing by means of specified coordinates is performed in the following sequence: first lay off the line-segment $Oa_x=x$ from the origin along the axis of abscissas (in our case, $x=5$); then draw through the point a_x a perpendicular to Ox, on which we lay off (noting signs) segments $a_xa=y$ (yielding a) and $a_xa'=z$ (yielding a'). The only thing left now is the profile projection of the point a''. Since the profile and vertical projections of the point must be situated on a single perpendicular to the z-axis, draw through a' a straight line $a'a''$ perpendicular to Oz.

Finally, the last question: At what distance from Oz should a'' be located?

Considering the coordinate parallelepiped (Fig. 10), the edges of which $a_za''=Oa_y=a_xa=y$, we conclude that the desired distance a_za'' is equal to y. Lay off the line-segment a_za'' to the right of Oz if $y>0$, and to the left, if $y<0$.

Let us now see what changes will occur on the orthographic drawing when the point changes its position in space.

For instance, let this point A (5, 4, 6) move along a line perpendicular to the V plane. In this motion, only one coordinate, y, indicating the distance from the point to the V plane, will change. The coordi-

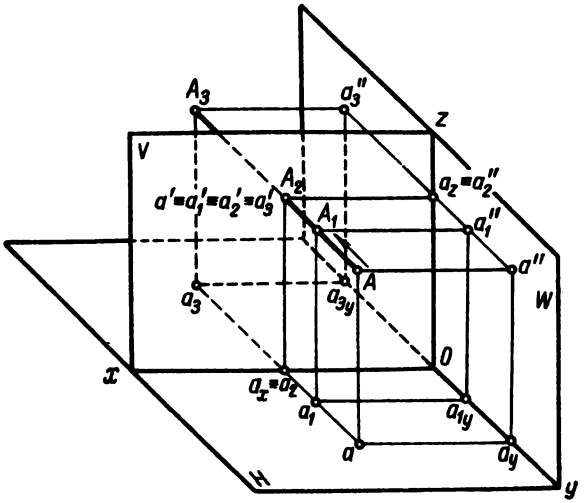


Fig. 12

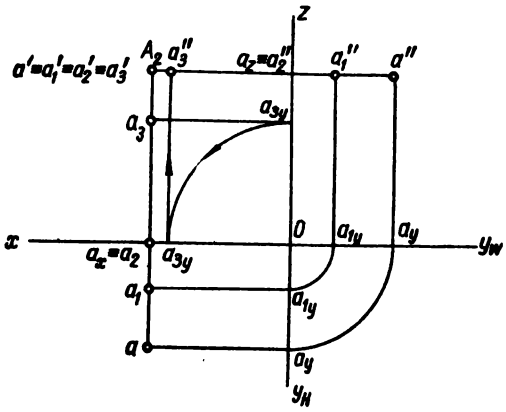


Fig. 13

nates x and z will remain unchanged, and the projection of the point defined by these coordinates (a') will not change its position.

Now take the projections a and a''. The former will begin to approach Ox, the latter, the z-axis. In Figs. 12 and 13, the new position of

the point is given by the notation A_1, a_1, a_{1y} and a_1''. When the point reaches the plane V ($y=0$), two of the three projections (a_2 and a_2'') will lie on the axes.

The point will move from the first octant to the second and will begin to recede from the V plane. The coordinate y will become negative and its absolute value will increase. The horizontal projection of the point in the second octant is located on the rear half-plane of H

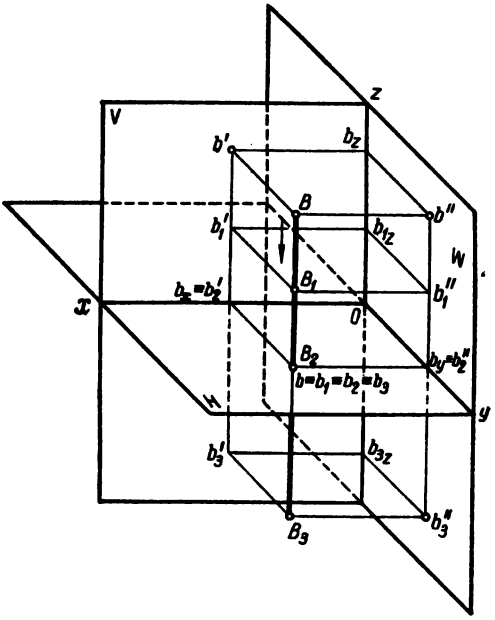


Fig. 14

and in the orthographic representation will be above the x-axis, the profile projection is located on the rear half-plane of W and orthographically will be to the left of the z-axis. As usual, $a_z a_z''=y$.

Let us examine another case of a point moving in space. Let the point B (Figs. 14 and 15) be in motion vertically downwards.

In this case, the z-coordinate will be variable, while the x- and y-coordinates and the horizontal projection of the point defined by them will remain unchanged. As long as the point is located within the first octant in its approach to the H plane, its vertical projection will tend towards the x-axis, and the profile projection towards the y-axis (see point B_1).

The point B_2 and its projections b_2, b_2' and b_2'' correspond to the case when $z=0$ and the point is located in the H plane. When the

point reaches the fourth octant, it will begin to recede from H. The z-coordinate will be negative and its absolute value will begin to increase. The vertical and profile projections of the point will as before be located on the common perpendicular to the z-axis in the lower half-planes of V and W.

Fig. 16 gives pictorial and orthographic representations of points located in different octants. Here, the number of any point corresponds to the octant in which it is located.

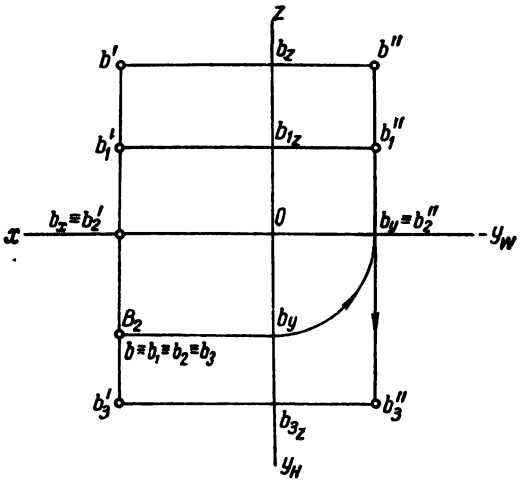


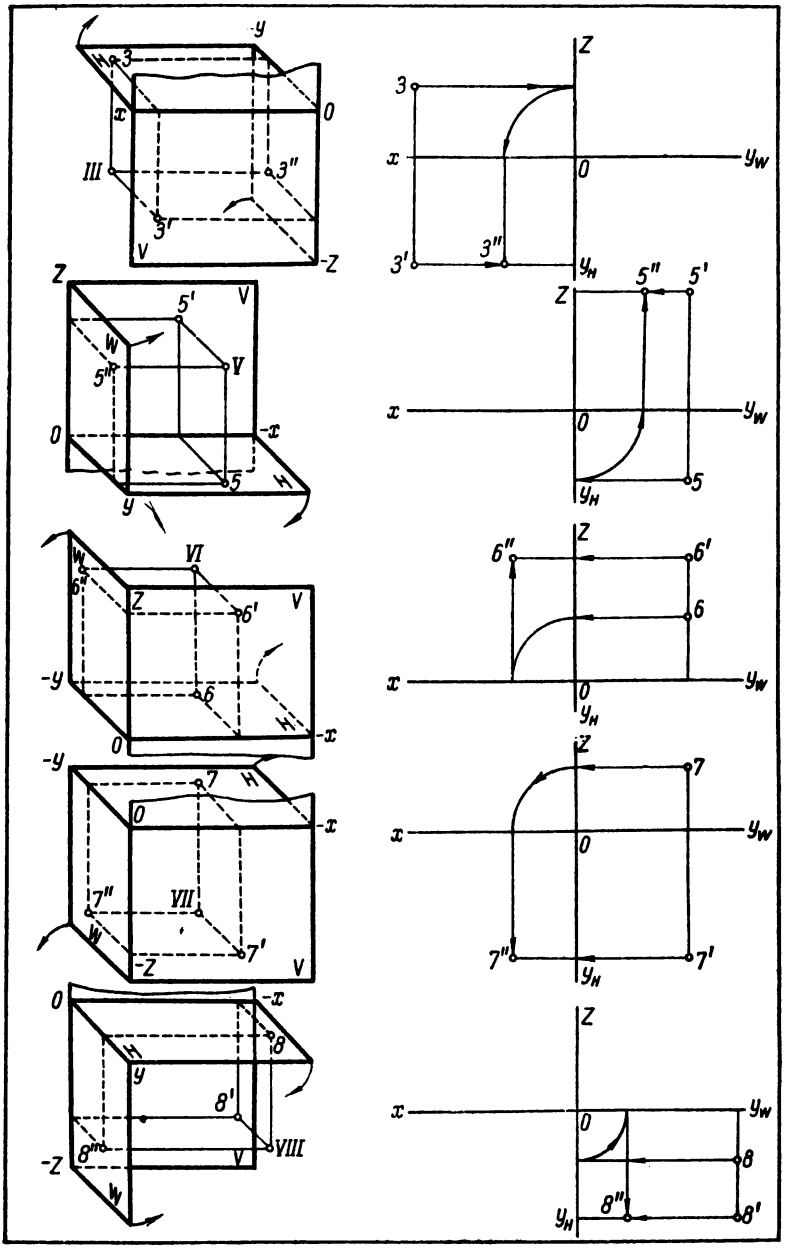
Fig. 15

In Fig. 16 and in subsequent figures we dispense with the literal labelling of the intersection points of coordinate axes with the lines of recall. This will somewhat simplify our drawings.

When an object is depicted in two projections and when the distance of the object from the W plane is of no significance, we shall drop the designation of the coordinate origin in orthographic representations.

Later, we shall encounter drawings without coordinate axes. This is a practical expedient for representing objects when only the representation of the object is significant and we are not interested in the specification of its position relative to the projection planes.

In this case the projection planes are determined with an accuracy only to within parallel translation (Fig. 17). The common practice is to translate them parallel to themselves so that all points of the object are above the H plane and in front of the V plane. Since the position of the x-axis is indeterminate, constructing an orthographic representation here does not involve rotation about the coordinate


y
3
H
x
0
III
3''
3'
V
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y
VI
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Z
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V
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-x
-y
7
H
0
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VII
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-Z
7'
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VIII
W
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Fig. 16

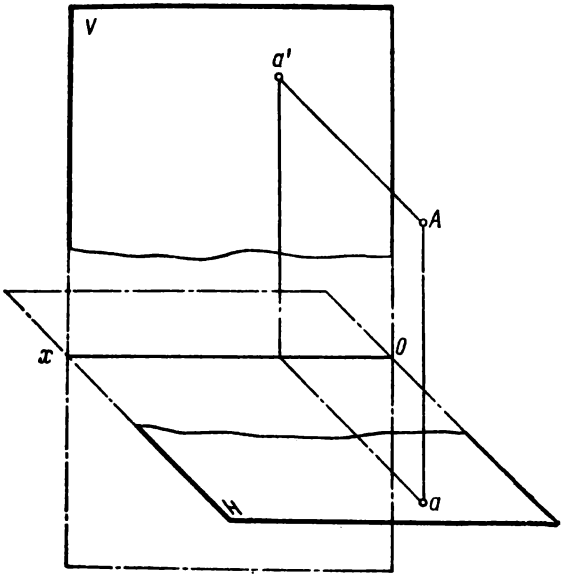


Fig. 17

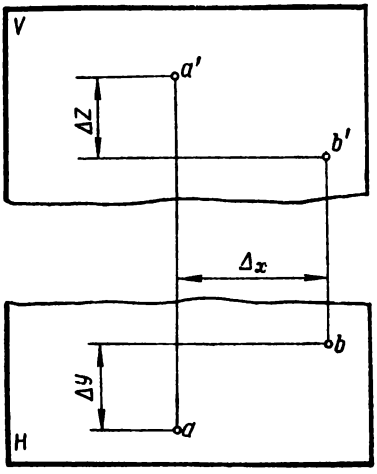


Fig. 18

axis. When passing to the orthographic drawing, the H and V planes are superposed in such a manner that unlike projections of points will lie on vertical line-segments.

The orthographic representation of points A and B without coordinate axis (or ground line) (Fig. 18) does not determine their positions in space but permits visualizing their relative orientation. Thus, line-segment Δx describes the displacement of point A relative to B in a direction parallel to the planes H and V. In other words, Δx indicates how far A lies to the left of B. The relative displacement of a point in a direction perpendicular to the V plane is determined by the line-segment Δy; that is, point A in our case is closer to the viewer than point B by a distance equal to Δy.

Finally, the line-segment Δz indicates how much higher A is than B.

Adherents of the axis-less study of descriptive geometry justly point out that one can dispense with coordinate axes in the solution of many problems. However it is not wise to reject them completely. Descriptive geometry is aimed not only at training the future engineer in the making of drawings but also in the solution of a variety of engineering problems, which prominently include problems of spatial statics and mechanics. This requires training to develop the ability of orienting an object relative to Cartesian coordinate axes. These skills will also be necessary in the study of such divisions of descriptive geometry as perspective and axonometry.

Most orthographic drawings of this text contain the coordinate axes. Thus not only is the shape of the object indicated, but also its position relative to the planes of projection.

CHAPTER II

THE STRAIGHT LINE

Sec. 7. Specifying a Straight Line

A straight line is, along with a point, the simplest element of any geometric figure; it is therefore subject to a detailed study by the methods of descriptive geometry.

The first question is: How do we define the position of a line in space?

The usual answer is: by two points. This is true, but it is not a full answer. In addition to two points, the position of a line may be determined by two intersecting planes, by two projections, by a point and

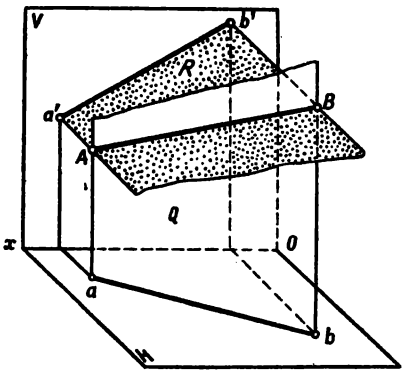


Fig. 19

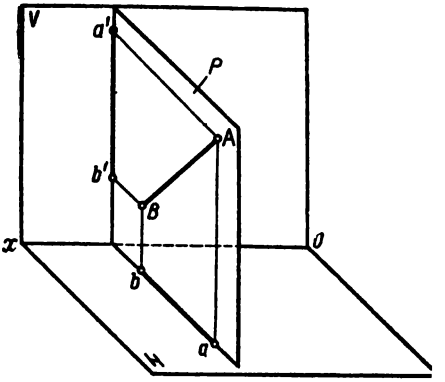


Fig. 20

angles of inclination to the projection planes, and by a point and the so-called direction cosines, by which are meant the cosines of angles that the line makes with the coordinate axes.

We shall not dwell on the first two methods of locating a straight line in space, since they are familiar to the reader who has studied elementary geometry; we shall show that *the position of a line is fully determined by two of its projections.*

Let there be two line-segments in the planes H and V: ab and $a'b'$ (Fig. 19). Through these line-segments pass planes Q and R perpendicular to the planes of projection. If the planes are not parallel and

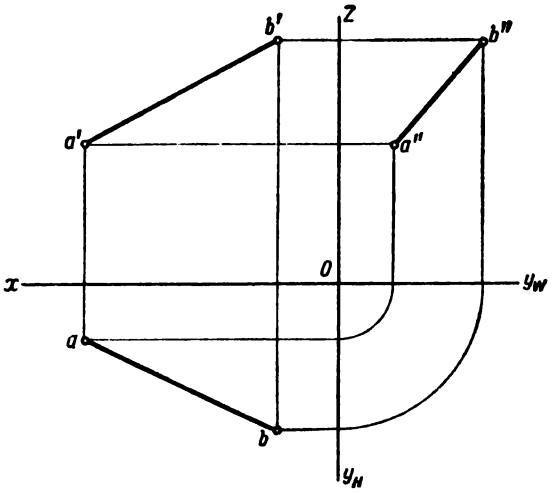


Fig. 21

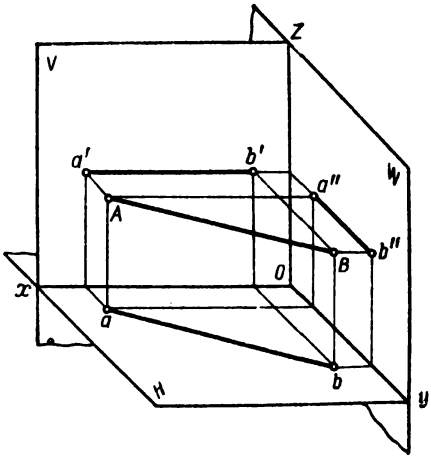


Fig. 22

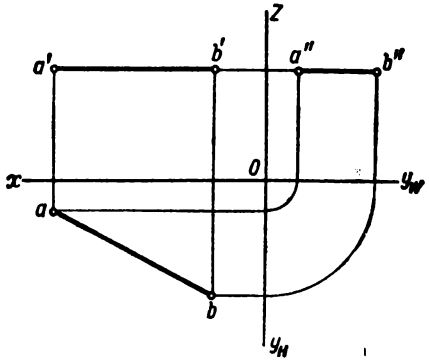


Fig. 23

intersect, the line of their intersection will be the line-segment AB (the only one in space) whose projections are ab and $a'b'$ (the former being the horizontal projection and the latter the vertical projection).

Planes Q and R may merge into a single plane P if ab and $a'b'$ are perpendicular to the x-axis (Fig. 20).

Here the straight line will be uniquely determined by its projections if each of them has any two points labelled with letters or numerals. If the labels are lacking, any line-segment lying in this plane may be taken as the desired line provided that AB is not parallel to H and AB is not parallel to V.

Sec. 8. Different Positions of a Line Relative to the Projection Planes

1. *An oblique line* (Fig. 19). This line is not parallel to any of the planes of projection. Orthographically (see Fig. 21), it is specified by the projections of two points: A and B. By connecting like projections of the points with straight lines, we get the projections of the line-segment.

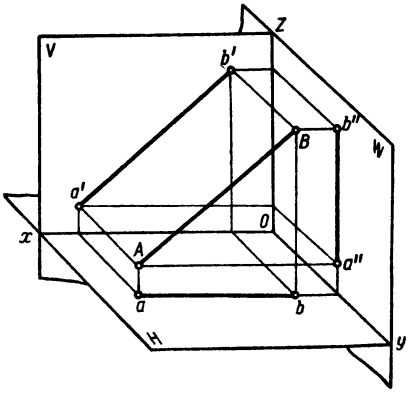


Fig. 24

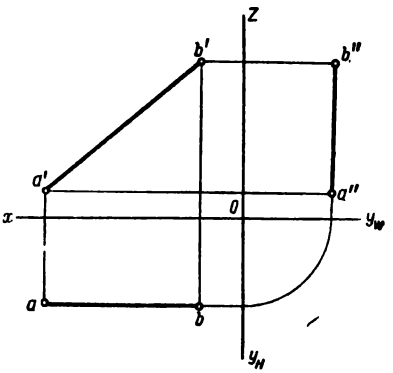


Fig. 25

2. *A line parallel to the horizontal plane of projection* (Figs. 22 and 23). These lines are called horizontal lines (or H parallels).

Since all points of the line are at the same distance from H, the equation $z_A = z_B$ holds for any pair of points of the horizontal line. Which means that orthographically the vertical projection $a'b'$ will be parallel to the x-axis, while the horizontal projection of the line-segment can occupy any position relative to the axis. The profile projection $a''b''$ is parallel to the y-axis.

3. *A line parallel to the vertical plane of projection* (Figs. 24 and 25). These lines are called vertical lines (or V parallels).

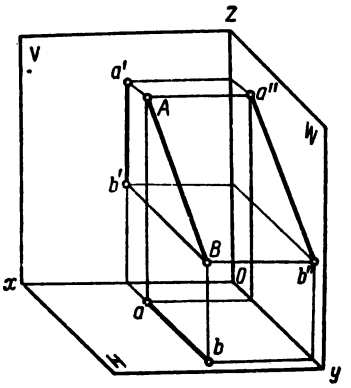


Fig. 26

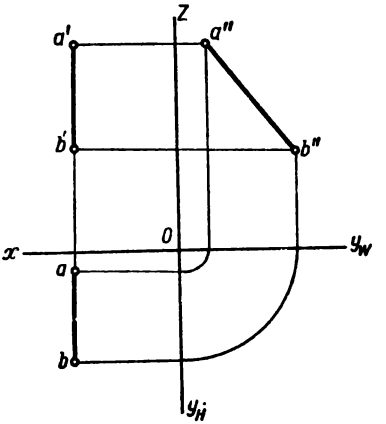


Fig. 27

Reasoning as in Item 2, we see that the horizontal projection of a vertical line is parallel to the x-axis, the vertical projection (in the general case) occupies an arbitrary position, and the profile projection is parallel to the z-axis.

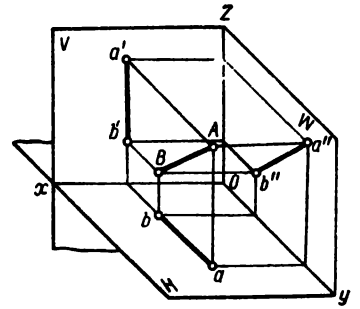


Fig. 28

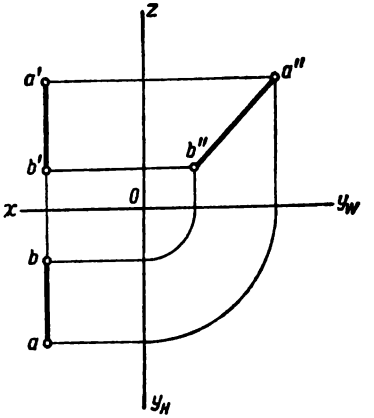


Fig. 29

4. *A line parallel to the profile plane of projection* (Figs. 26 and 27). These lines are called profile lines. In this case, for any two points of a profile line we have the equation $x_A = x_B$, which means the horizontal

and vertical projections of the line will be perpendicular to the x-axis.

We distinguish between ascending and descending profile lines. The former rise upon recession from the viewer, while the latter

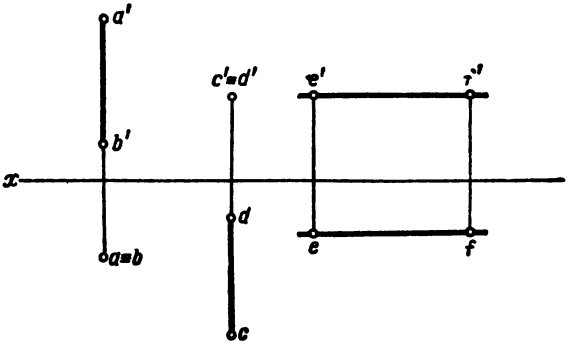


Fig. 30

descend. Figs. 26 and 27 illustrate a line-segment of an ascending profile line; Figs. 28 and 29 depict a segment of a descending line.

5. *Lines perpendicular to the plane of projection.* Fig. 30 shows the projections of segments of three lines: AB, CD and EF, which are re-

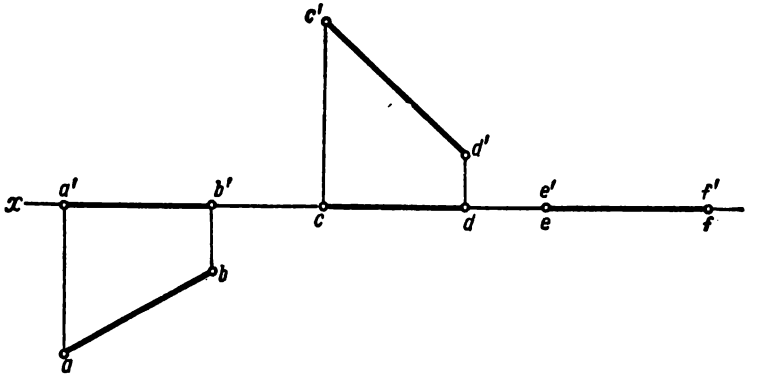


Fig. 31

spectively perpendicular to H, V and W. A straight line is projected into a point on a plane to which it is perpendicular.

6. *Lines in the planes of projection.* A characteristic feature of line-segments of this kind is that two of the three projections of these lines always lie on the respective coordinate axes. Fig. 31 shows two

projections of line-segments: AB lying in H, CD in V and EF coincident with the x-axis.

In conclusion it is worth noting that in cases when a straight line is parallel to some projection plane, the line is projected onto this plane full length.

Sec. 9. Dividing a Segment of a Straight Line in a Given Ratio

On the basis of proved properties of parallel projection, we may assert that if point C divides a line-segment AB in the ratio $m : n$, then the projections of C also divide the projection of the segment in the same ratio. Put briefly, the ratio of the segments of a straight

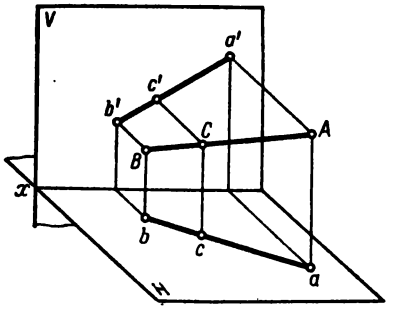


Fig. 32

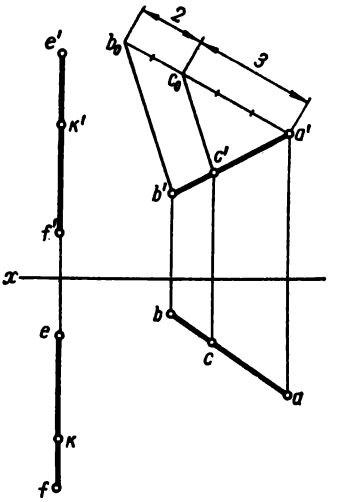


Fig. 33

line is equal to the ratio of their projections (Fig. 32):

$$\frac{AC}{CB} = \frac{ac}{cb} = \frac{a'c'}{c'b'}$$

In Fig. 33 the line-segment AB is divided in the ratio 3 : 2.

Sec. 10. Determining the Length of a Line-segment and the Angles of Inclination of a Line to the Planes of Projection

As will be seen from Fig. 34, the length of the line-segment AB may be determined from the right triangle ABb_1 in which the side $Ab_1 = ab$ (the projection of AB on H), and the side Bb_1 is equal to Δz, the difference of the distances of the points A and B from the H plane.

Angle α in this triangle determines the slope angle of AB to the H plane.

If in place of H we take V, the length of the line-segment AB may be determined in similar fashion from the right triangle ABa_1' (Fig. 35) where: the side Ba_1 is equal to the projection $a'b'$, the sec-

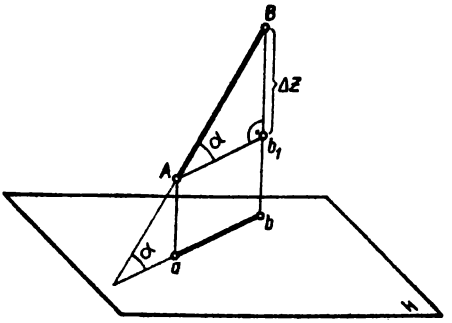


Fig. 34

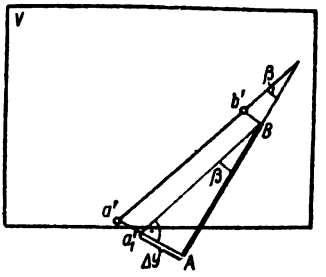


Fig. 35

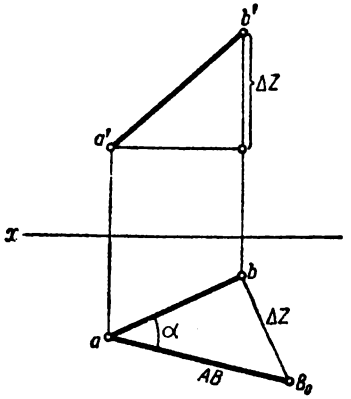


Fig. 36

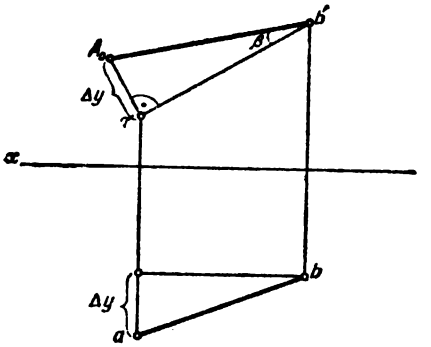


Fig. 37

ond side Aa_1' is equal to Δy, or the difference of the distances of the points A and B from the V plane. The angle β in the same triangle ABa_1' determines the slope angle of the line-segment AB to the V plane.

Figs. 36 and 37 illustrate (orthographically) finding the length (aB_0 and $a'B_0$) of a line-segment and the slope angles to the planes H (angle α) and V (angle β).

Sec. 11. Mutual Positions of a Point and a Line-segment

If a point C lies on a line-segment AB, the projections of the point must lie on like projections of this line-segment (see Figs. 32 and 33). The converse assertion—if the projections of a point lie on like projections of the line-segment, then the point also lies on this line—does not always hold.

Thus, without supplementary constructions it is impossible to establish the relative positions of a point K and a profile line EF (Fig. 33) from two projections.

To evaluate the mutual positions of a point and a line-segment in this special case, it is necessary to construct their profile projections or to verify the equality

$$\frac{e'k'}{k'f'} = \frac{ek}{kf}$$

If the equality is fulfilled, that is, if the projections of the point divide the corresponding projections of the line-segment in the same ratio, then the point K will lie on the line-segment EF.

Sec. 12. Traces of a Line

The trace of a line is the point in which the line intersects a projection plane.

In the general case the line may intersect all three planes of projection and have three traces:

the *horizontal* trace—M (m, m', m''), the point of intersection of the line with the H plane,

the *vertical* trace—N (n, n', n''), the point of intersection of the line with the V plane, and

the *profile* trace—T (t, t', t''), which is the intersection with the W plane.

Let us first examine the line-segment AB in a system of two planes of projection and let us find its horizontal and vertical traces.

In Fig. 38, these traces are defined as points in which a line-segment intersects its projections. Each trace, being a point which belongs at one and the same time both to a given line-segment and to one of the projection planes coincides with its projection of the same name. Thus, M coincides with m, and N with n'. Projections different from the given trace will he located on the x-axis; that is, the vertical projection m' of the horizontal trace M and the horizontal projection n of the trace N should lie on the x-axis, and these will be points at which the axis intersects the appropriate projections of the given line-

segment. The intersection of ab and the x-axis determines n and the intersection of $a'b'$ and the x-axis yields m'

The foregoing peculiarities in the positions of projections of traces permit formulating the following rules for an orthographic construction of traces:

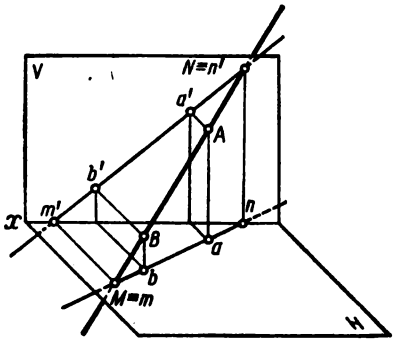


Fig. 38

1. To construct the horizontal trace M of a line-segment, prolong its vertical projection to intersection with the x-axis and at that point erect a perpendicular to the axis to intersection with the horizontal projection of the line.

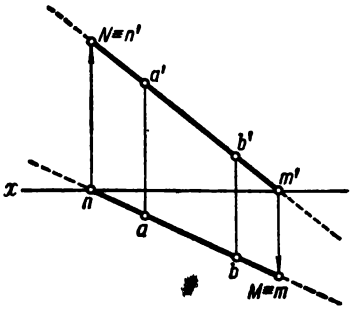


Fig. 39

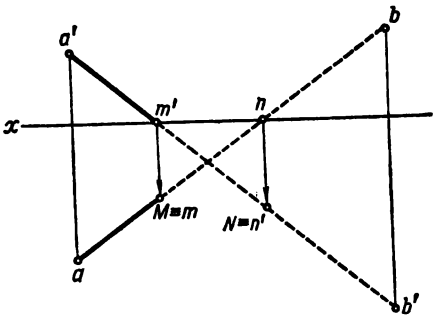


Fig. 40

2. To construct the vertical trace N of a line-segment, from the point of intersection of its horizontal projection with the x-axis erect a perpendicular to the axis to intersection with the vertical projection of the line.

In Figs. 39 and 40 these rules yield traces of straight lines.

Traces of lines are points in which a line passes from one octant into another; they enable us to indicate the visibility of the line. The visible part of a line is that part which is located within the limits of the first octant. Projections of a line corresponding to the visible part will be depicted by solid lines, the respective hidden parts are indicated by dashed lines.

We consider especially the construction of traces of a profile line, the projections of which on H and V are perpendicular to the x-axis (Figs. 41 and 42).

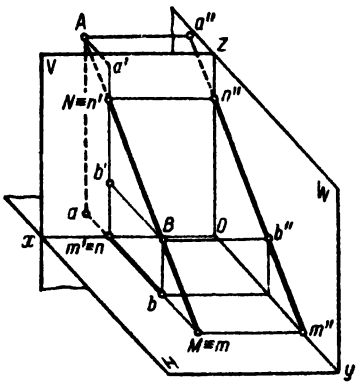


Fig. 41

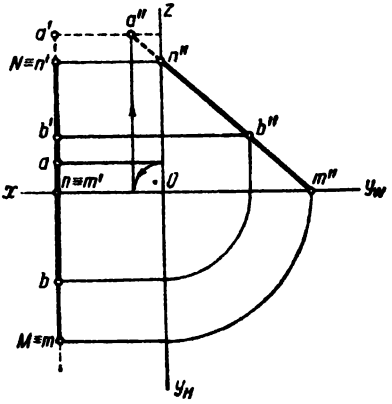


Fig. 42

It is not possible to determine directly its traces from the horizontal and vertical projections, since all auxiliary lines perpendicular to the x-axis merge into a single line with the projections of the given line. In that case, one has to resort to the profile projection of the line, whose intersection with the y-axis determines m''—the profile projection of the horizontal trace; the intersection with the z-axis determines n'' which is the profile projection of the vertical trace. From then on the construction of the traces themselves and the two other projections is obvious from Figs. 41 and 42.

We now consider the construction of traces of a line-segment in a system of three planes of projection.

The horizontal and vertical traces of a line will be determined by the same rules as those given above.

The only added requirement will be to indicate also the profile projections of these points.

The profile trace T in Fig. 43 is found as the point of intersection of the line MN with its profile projection. The trace T will coincide with its like profile projection.

The two other projections of this trace, like any point situated in the plane W, will lie on the y- and z-axes. Since T is a point on this given line, its projections t and t' will lie on like projections of the line.

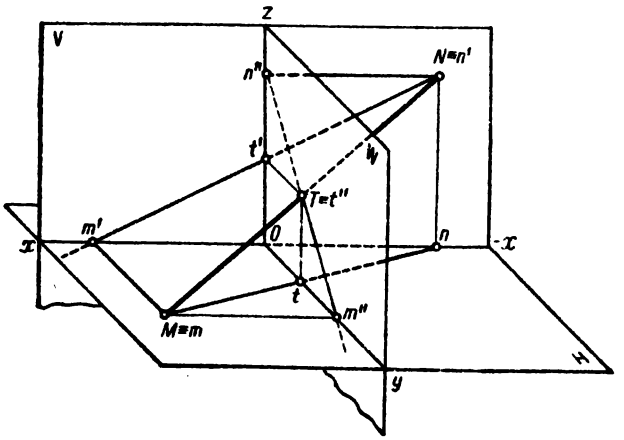


Fig 43

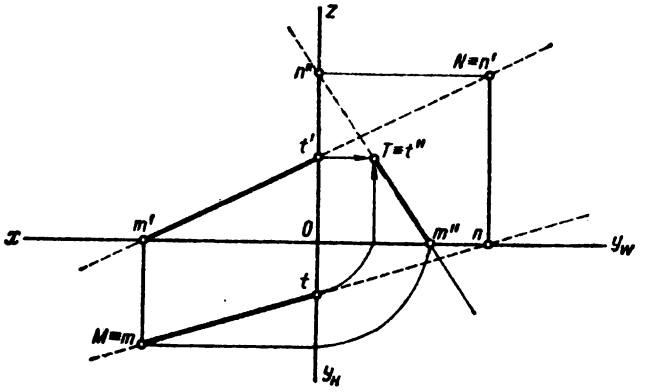


Fig. 44

On the orthographic drawing given in Fig. 44, the profile trace of the line and its projections are determined in the following manner:

1 Note the point of intersection of the vertical projection of the line with the z-axis: point t'.

2. Through this point draw a perpendicular to the z-axis to intersection with the profile projection of the line. The point obtained will be the desired trace T with which t'' coincides.

3. The horizontal projection t will be determined by intersection of the horizontal projection of the line with the y-axis.

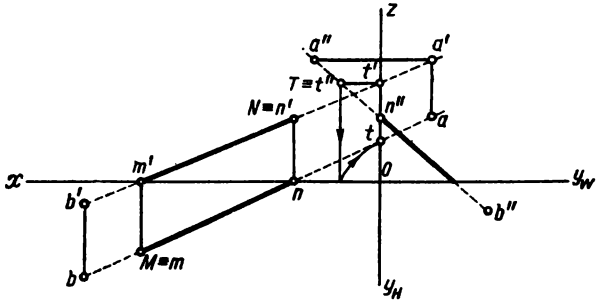


Fig. 45

Fig. 45 portrays yet another case of determining traces of the line-segment AB that passes through the sixth, second, first and fourth octants.

Sec. 13. The Relative Positions of Two Straight Lines

1. *Intersecting lines.* In this case the lines have a common point, the projections of which, e and e', lie on a single perpendicular to the x-axis (Fig. 46). Thus, *if line-segments intersect, the points of intersection of their like projections must lie on a single perpendicular to the coordinate axis.*

2. *Parallel lines.* In Sec. 3 it was demonstrated that the projections of parallel lines on any plane are parallel. This property of parallel projection remains true for orthogonal projections as well; that is if AB is parallel to CD, then ab is parallel to cd, $a'b'$ to $c'd'$ (Fig. 47).

In the general case, the converse proposition holds as well: if in an orthographic representation like projections of lines are parallel, then the lines in space are also parallel. Indeed, projecting planes drawn through the projections of straight lines will intersect along mutually parallel lines.

Lines parallel to one of the projection planes represent a special case. For instance, vertical and horizontal projections of profile lines are parallel, but it is necessary to refer to the third projection to evaluate their relative positions and draw a final conclusion. The profile projections of the lines AB and CD constructed in Fig. 48 permit, us to conclude that AB and CD are not parallel.

3. *Skew lines. If the lines do not intersect and are not parallel to each other, then the points of intersection of their like projections do not lie on a single perpendicular to the axis* (Figs. 49 and 50).

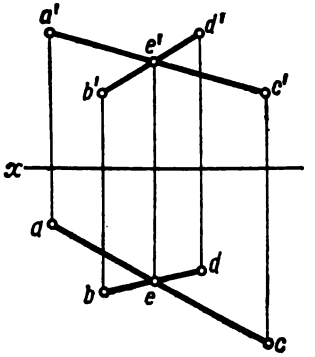


Fig. 46

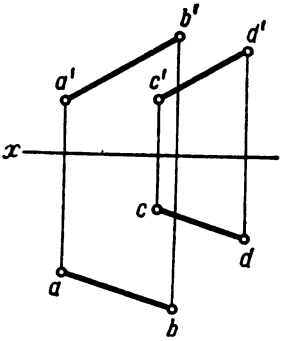


Fig. 47

To the point of intersection of the vertical projections of the lines there correspond two points E and F, of which one belongs to line AB and the other to CD. Their vertical projections coincide solely because

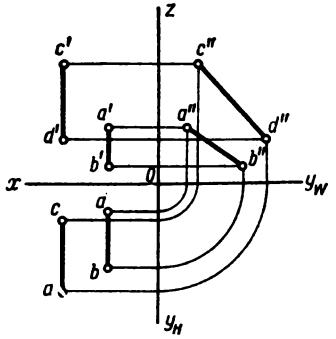


Fig. 48

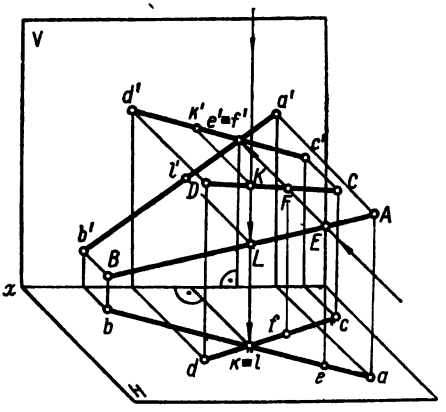


Fig. 49

in space both points E and F are situated on a common perpendicular to the V plane. The horizontal projection of this perpendicular, denoted by arrows (Fig. 50), enables us to determine which of the two points is closer to the viewer.* In our case, it is E which lies on AB.

* We assume that the straight lines are located between the viewer and the projection plane.

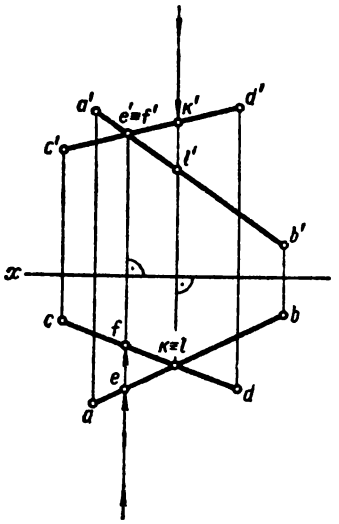


Fig. 50

Hence, the line AB passes (here) in front of CD.

To the point of intersection of the horizontal projections there also correspond two points K and L located on different lines. Their horizontal projections coincided because both points in space were on a single perpendicular to the H plane.

The vertical projection (indicated by arrows in Fig. 50) of this perpendicular answers the question: Which of the two points is higher?

As may be seen from the drawing, the point K is above L. Hence, the line CD passes above AB.

Sec. 14. Mutually Perpendicular Lines

If two lines intersect at a right angle, their projections will, in the general case, form an angle not equal to 90°.

For a right angle to project into its true magnitude, it is sufficient that one of its sides be parallel to the projection plane.

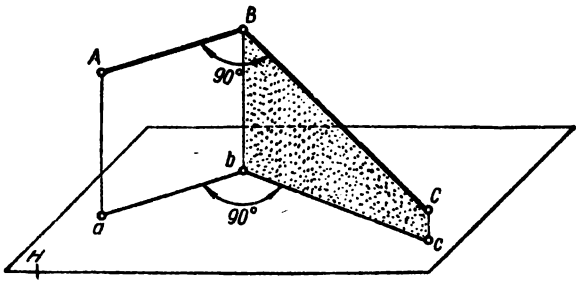


Fig. 51

Indeed, let side AB of the right angle ABC be parallel to the H plane. We have to prove that its projection, angle abc, is equal to 90° (Fig. 51).

The line AB is perpendicular to the plane $BCcb$, since AB is perpendicular to two lines of this plane, BC and Bb, which pass through the base B.

AB and its projection *ab* are two parallel lines, and therefore *ab* is also perpendicular to the plane *BCcb*. Consequently, *ab* is perpendicular to *bc*.

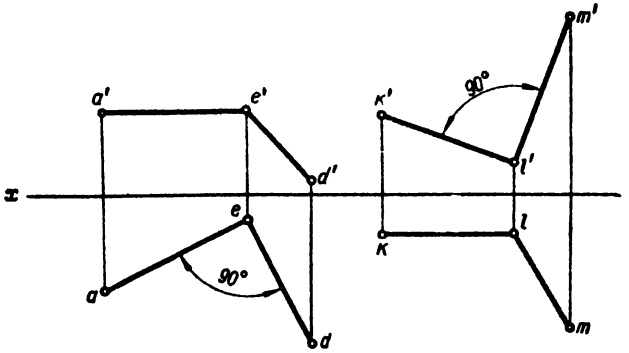


Fig. 52

The foregoing enables us to state that the angles indicated in Fig. 52 are projections of right angles. In the first, the side *AE* is parallel to the plane *H* (see Sec. 8), and in the second, the side *KL* is parallel to *V*.

CHAPTER III

THE PLANE

Sec. 15. Ways of Specifying a Plane. Traces of a Plane

The position of a plane in space may be determined by: (1) three points not lying on one line (Fig. 53); (2) a line and a point not lying on the line (Fig. 54); (3) two intersecting lines (Fig. 55); (4) two parallel lines (Fig. 56).

Each one of these methods of specifying a plane permits us to pass to any one of the others.

It is convenient to judge the position of a plane relative

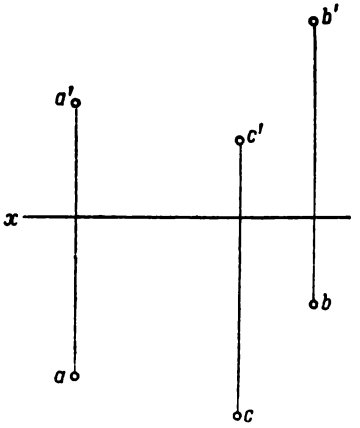


Fig. 53

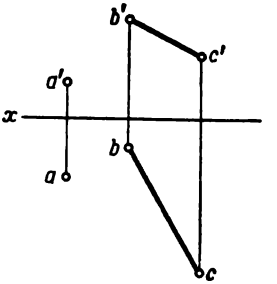


Fig. 54

to the projection planes from its traces, which are lines along which the given plane intersects the planes of projection. An oblique plane (not perpendicular to any of the projection planes) has three traces: horizontal, vertical and profile.

Let us agree to designate the planes in space by the letters P, Q, R, ..., and their traces, by the same letters with the subscript H for the horizontal trace, P_H, Q_H, R_H, ..., the subscript V for the vertical trace, P_V, Q_V, R_V, ..., and the subscript W for the profile trace, P_W, Q_W, R_W, ...

The oblique plane P is shown in Fig. 57. The traces P_H, P_V, P_W intersect in pairs on the axes in points P_x, P_y and P_z, which are called vanishing points. These points may be regarded as the vertices

of trihedral angles formed by the given plane and two of the three projection planes.

Each of the traces of a plane coincides with its like projection, and the other two (different projections) lie on the axes. For example, a horizontal trace of a plane coincides with its horizontal projection, but its vertical projection lies on the x-axis, and its profile projection lies on the y-axis. Unlike projections of each trace remain unlabelled when represented orthographically.

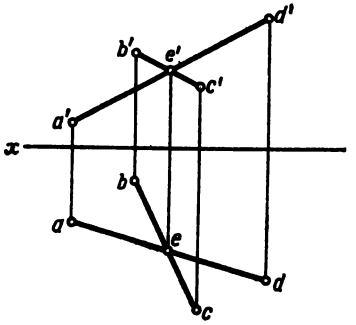


Fig. 55

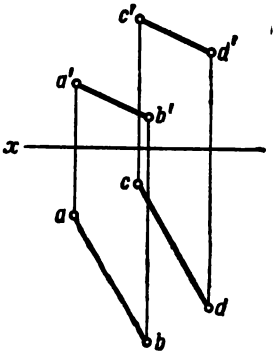


Fig. 56

Bringing the horizontal and profile planes to coincidence with the vertical plane, we get a representation of the plane P by its traces in the orthographic representation (Fig. 58).

Note that the vanishing point P_y is orthographically depicted twice as a point belonging to the y-axis.

Any two traces of a plane fully determine the position of the plane in space (as two intersecting lines). The third trace of the plane can always be constructed from the two given ones. In Fig. 59, the profile trace P_W is constructed with the aid of the vanishing points P_y and P_z, in which the given traces P_H and P_V intersect the y-axis and z-axis, respectively. When representing the traces of a plane orthographically, it is necessary to note their visibility. Only that part of each of the traces is visible that lies in the first octant. For example, a segment of the profile trace P_yP_z (see Figs. 59, 60) is hidden because the part of plane W on which it lies is outside the first octant.

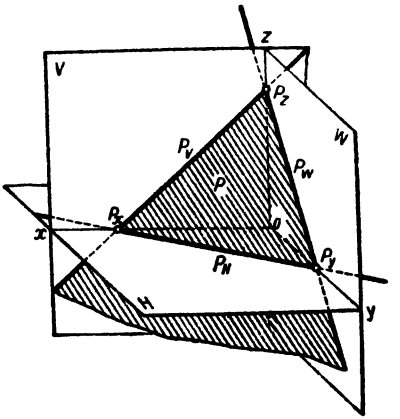


Fig. 57

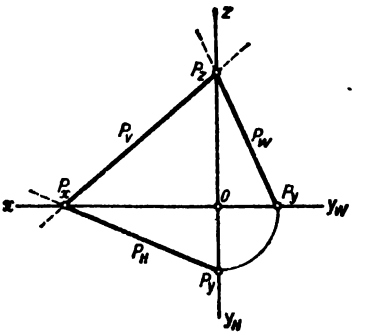


Fig. 58

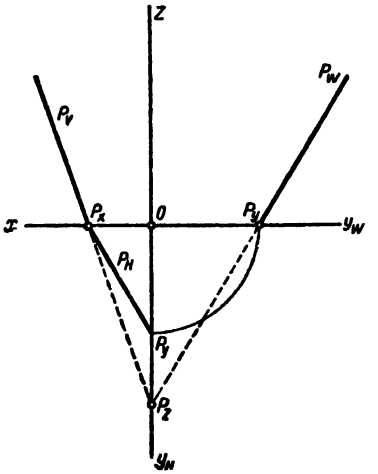


Fig. 59

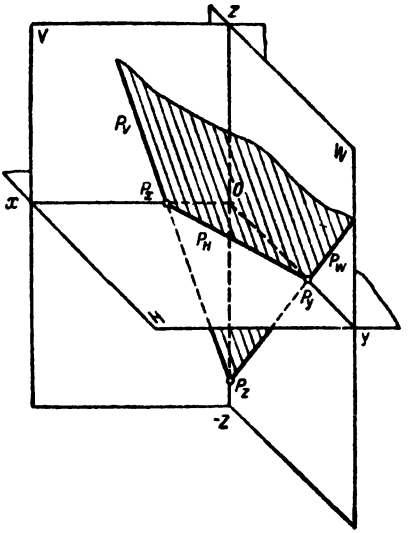


Fig. 60

Sec. 16. Various Positions of a Plane Relative to the Projection Planes

1. Plane Q, which is perpendicular to H, is a *horizontal projecting plane* (Figs. 61 and 62).

The distinguishing feature of this plane in an orthographic representation is the perpendicularity of the vertical trace Q_V to the x-axis. The angle between Q_V and the x-axis will be a right angle because Q_V is a line of intersection of two planes that are perpendicular to H. For the same reason, the profile trace Q_W forms a right angle with the y-axis.

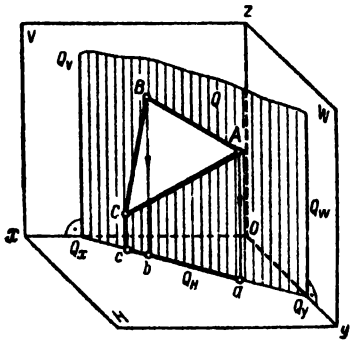


Fig. 61

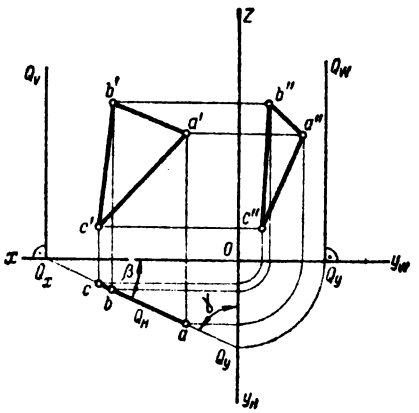


Fig. 62

The horizontal projections of all points and any figures lying in this plane will coincide with the horizontal trace Q_H. For instance, the horizontal projection of the triangle ABC situated in plane Q is a straight line coinciding with Q_H (see Figs. 61 and 62).

The angles β and γ which Q makes with the planes V and W project true size onto H.

2. The plane R (which is perpendicular to the V plane) is a *vertical projecting plane* (Figs. 63 and 64).

The characteristic feature of this plane in an orthographic representation is the perpendicularity of the horizontal trace R_H to the x-axis. The angle between R_H and the x-axis will be a right angle by virtue of the fact that R_H is the line of intersection of two planes that are perpendicular to V. For this same reason, the profile trace R_W makes a right angle with the z-axis.

The vertical projections of all points and any figures lying in this plane will coincide with the vertical trace. To illustrate, the vertical

projection of the triangle ABC, which lies in the plane R, is a straight line that coincides with R_V.

The angles α and γ which the R plane makes with H and W project true size onto the V plane.

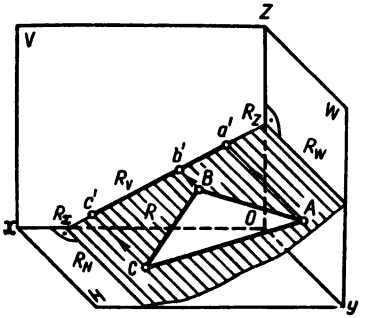


Fig. 63

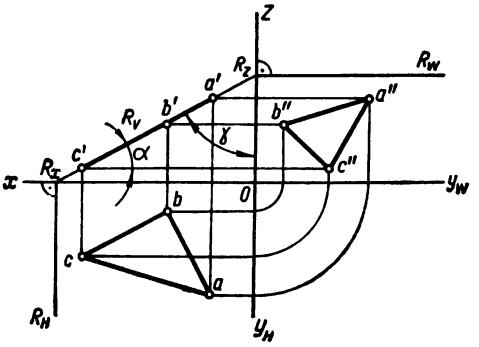


Fig. 64

3. The S plane, which is perpendicular to the W plane, is the *profile projecting plane.*

Figs. 65 and 66 show the special case when the profile projecting plane passes through the x-axis and bisects the angle between the H and V planes. The horizontal and vertical traces of this bisector plane

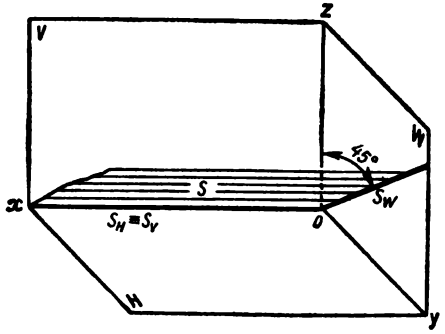


Fig. 65

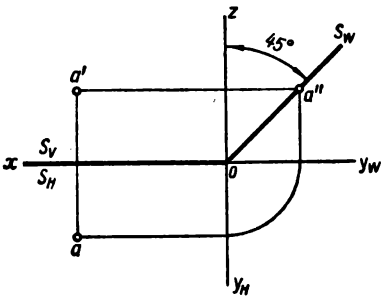


Fig. 66

coincide with the x-axis, and the profile trace makes an angle of 45° with the z-axis.

The profile projections of all points of this plane coincide with the profile trace.

4. The plane T, which is parallel to the H plane (Figs. 67 and 68), is called a *horizontal plane.*

Since the z-axis is perpendicular to the T plane, it makes right angles with its vertical and profile traces. As a result, the vertical trace of the T plane is parallel to the x-axis, and this plane has no horizontal trace.

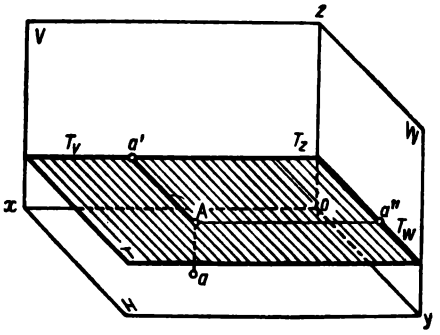


Fig. 67

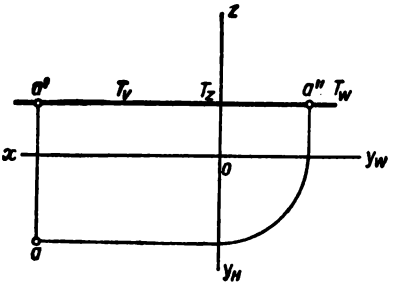


Fig. 68

The T plane intersects V and W at right angles and thus combines the properties of vertical and profile projecting planes. For this reason, any figure lying in this plane is projected on V and W in the form of lines that coincide with like traces. The horizontal projection of this figure will be true size.

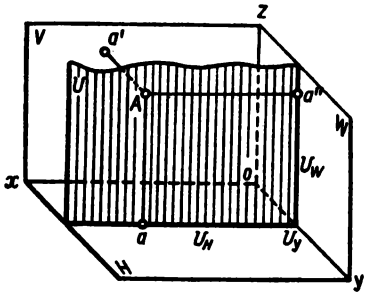


Fig. 69

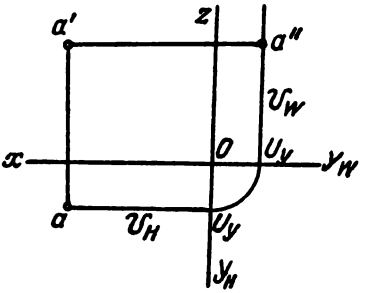


Fig. 70

5. The plane U, which is parallel to the V plane (Figs. 69 and 70) is called a *vertical plane.*

Since the y-axis is perpendicular to the U plane, it makes right angles with its horizontal and profile traces. The horizontal trace of the U plane is parallel to the x-axis; this plane has no vertical trace.

The U plane intersects H and W at right angles and thus has the properties of horizontal and profile projecting planes. For this reason,

any figure lying in this plane is projected on the H and W planes in the form of lines that coincide with like traces.

The vertical projection of this figure will be true size.

We conclude this section by stressing once again the basic property of projecting planes: *if a line or figure lies in a plane perpendicular to some plane of projection, the line or figure will be projected onto this plane in the form of a line that coincides with the like trace of the projecting plane.* The projection of a line lying in this plane may, in a special case, be a point, but this point will likewise be located on the like trace of the projecting plane.

Sec. 17. Constructing Traces of a Plane

Every trace of a plane is a line for the construction of which we have to know either two points, or one point and a direction.

For the two points used to determine the position of the trace of a plane we can take like traces of two lines lying in the plane.

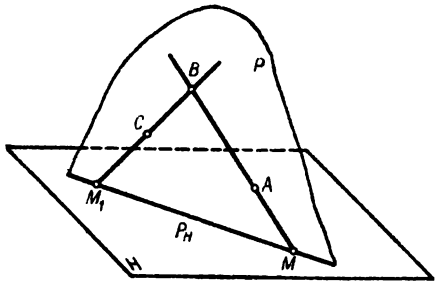


Fig. 71

Fig. 71 shows the construction of the horizontal trace of plane P by means of like (horizontal, that is) traces of the lines AB and BC which locate the plane P.

Figs. 72 and 73 illustrate the construction of the traces of a plane given by three points. The horizontal trace of the plane is determined by the horizontal traces M and M_1 of lines AB and BC. The vertical trace is constructed by means of the like traces N and N_1 of lines AB and AC. It may be noted that P_V might have been constructed by means of the vertical trace of one of the lines and the vanishing point P_x.

Finally, the vanishing points of the traces P_y and P_z enable us to construct the third trace, the profile trace P_W of the plane.

If the vanishing points P_y and P_z were outside the drawing, we would have to determine the profile traces of two straight lines of the plane in order to construct P_W.

We now consider the construction of traces of a plane when one of two lines defining the position of the plane is parallel to the x-axis (Fig. 74). In this case, the P plane and its traces P_H and P_V will be parallel to the x-axis. Thus, to construct P_H and P_V it is necessary to find one point in each through which they will pass. They are the traces M and N of the second line BC.

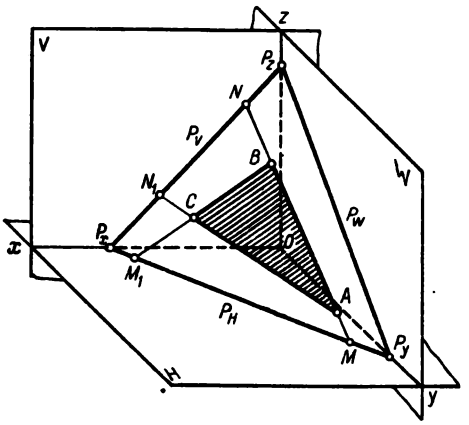


Fig. 72

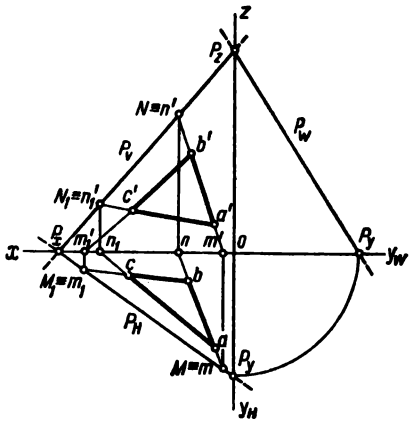


Fig. 73

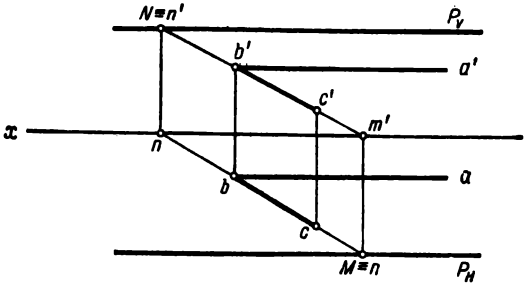


Fig. 74

Sec. 18. Lines and Points in a Given Plane

If the line AB (in plane P) is extended to intersection with the traces of the plane in points M and N, the points thus obtained will be the traces of the given line on the planes of projection (Fig. 75). Consequently, *if a line is contained in a plane, its traces lie on like traces of the plane.*

A line may belong to a plane and be parallel to one of its traces. Then the line must have a common point with the other trace of the plane. (Such lines are considered in the next section.)

In an orthographic representation a plane is not always specified by traces. Then how do we draw straight lines in the plane while dispensing with the construction of traces in the plane?

In Fig. 76, a plane is given by two intersecting lines. To draw an arbitrary straight line in the plane, we take two points I and II, the

first on AB and the second on BC. Joining the like projections of these points, we have the projections 1-2 and $1'$-$2'$ of the line I-II. The line thus constructed belongs to the given plane since they have two points in common.

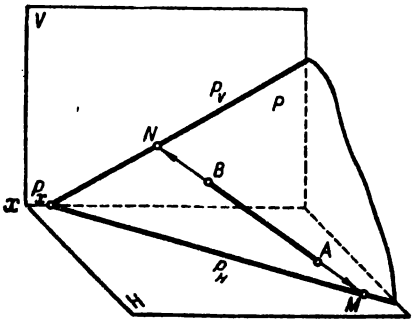


Fig. 75

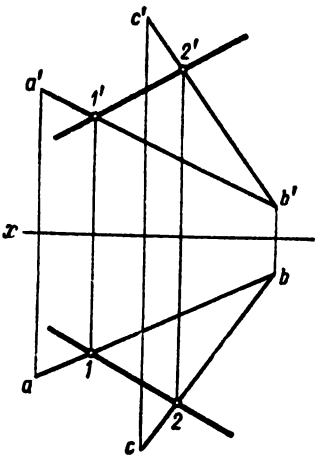


Fig. 76

Another frequently encountered graphical operation performed in a plane consists in the construction of separate points.

We solve the following problem:

Given: a point K in a plane specified by two parallel lines AB and CD. Required: to determine its horizontal projection if the vertical projection k' is known (Fig. 77).

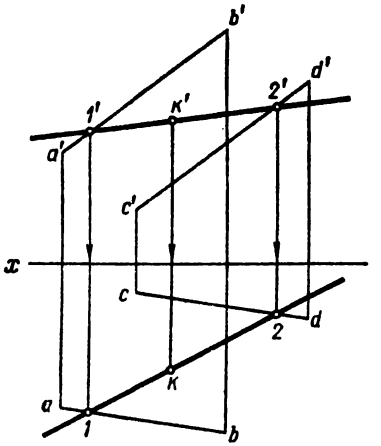


Fig. 77

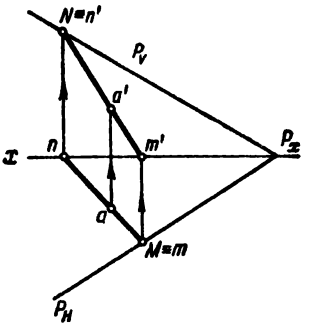


Fig. 78

Let us visualize a certain line *I-II* drawn through *K* in the plane. The vertical projection, *1'-2'*, must pass through the given projection *k'* of point *K*.

The second projection of this line is determined by means of the horizontal projections of points *I* and *II* in which an auxiliary line

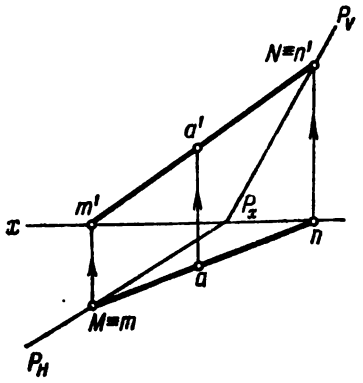


Fig. 79

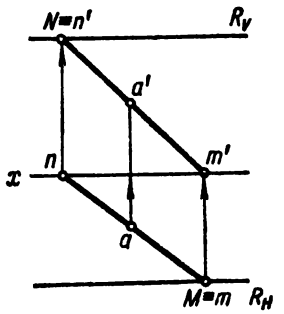


Fig. 80

intersects the two given lines. The required horizontal projection of *K* lies on a like projection of the line *I-II*. Figs. 78-80 illustrate the solution of similar problems for cases when the plane is specified by traces. The solution is also obtained by means of the line *MN*, which in each case was drawn in the plane through the point *A*.

In these orthographic representations, the horizontal projections of the points are assumed to be given.

The solution of a whole series of problems involves the construction of points belonging to a particular surface. In such cases, the usual procedure is to draw a certain line on a given surface and then determine the position of the required point on this line.

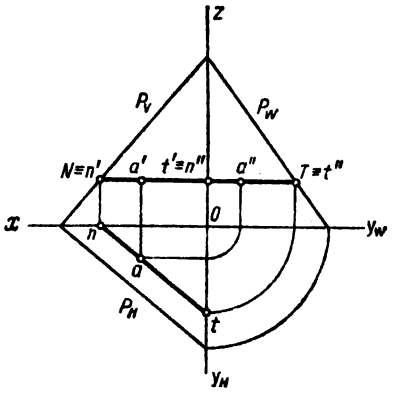


Fig. 81

Sec. 19. Principal Lines of a Plane

Of the straight lines that may be situated in a given plane, of special importance are the lines of four directions:

1. *H* principal lines (also called *H* parallels or horizontal lines) are lines lying in a given plane and parallel to the horizontal plane of projection (Fig. 81).

2. *V* principal lines (also called *V* parallels or vertical lines) are lines lying in a given plane and parallel to the *V* plane (Fig. 82).

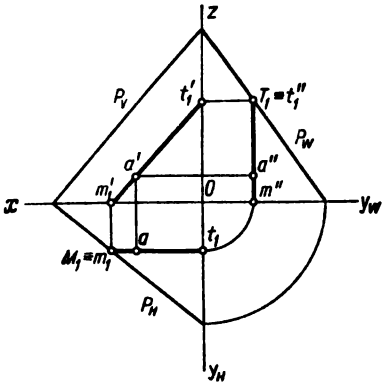


Fig. 82

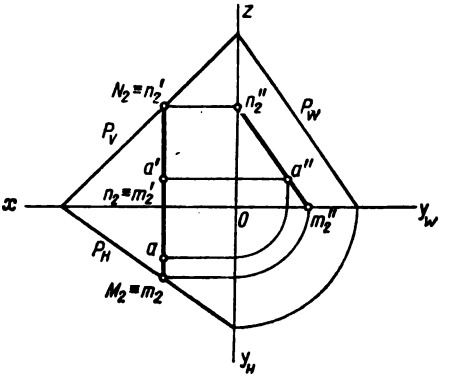


Fig. 83

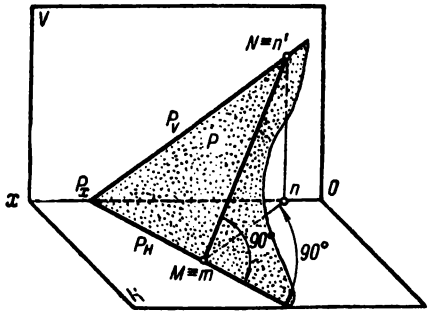


Fig. 84

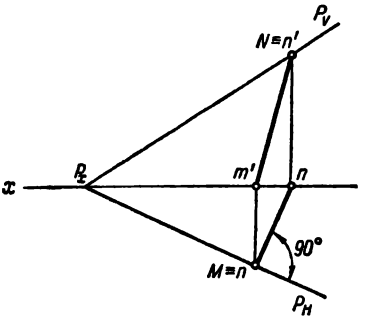


Fig. 85

3. Profile lines are lines lying in a given plane and parallel to the *W* plane (Fig. 83).

4. The steepest lines are lines drawn in a plane perpendicular to horizontal lines (Figs. 84 and 85).

The foregoing lines are known as the principal lines of a plane. The first three (shown in Fig. 86) determine the directions of the appropriate traces of the plane: the horizontal line NT is parallel to P_H, the vertical line M_1T_1 is parallel to P_V and the profile line N_2M_2 is parallel to P_W.

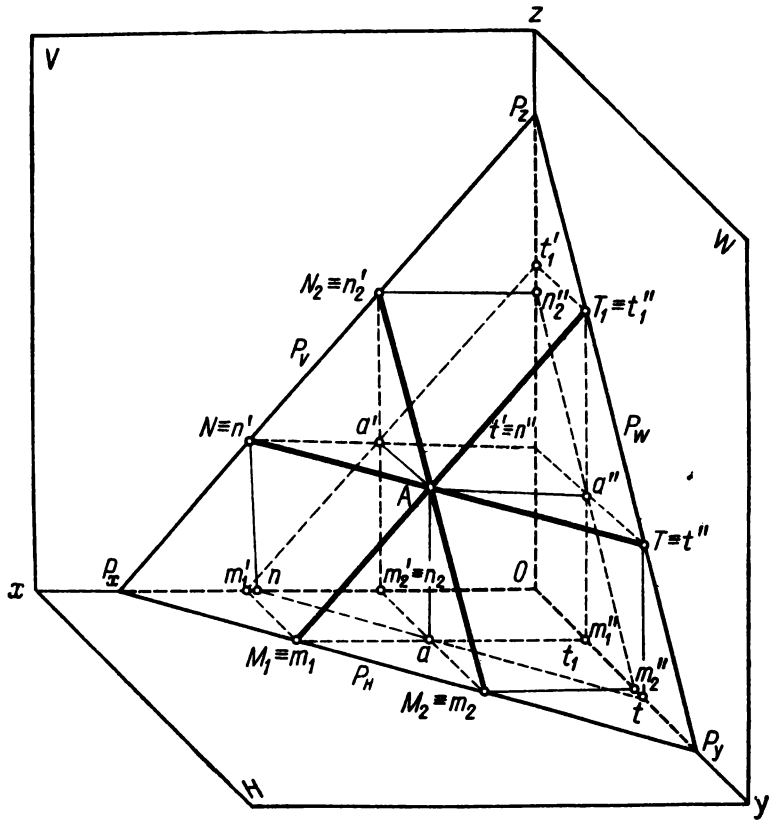


Fig. 86

Any plane can contain an infinite number of principal lines. All the lines of any one of the four directions form a plane pencil of parallel straight lines, which means that all the horizontal lines of a plane are parallel, all the vertical lines of a plane are parallel, and so forth.

Principal lines of a plane	Projections		
	horizontal	vertical	profile
Horizontal line	parallel to P_H	parallel to x-axis	parallel to y-axis
Vertical line	parallel to x-axis	parallel to P_V	parallel to z-axis
Profile line	parallel to y-axis	parallel to z-axis	parallel to P_W

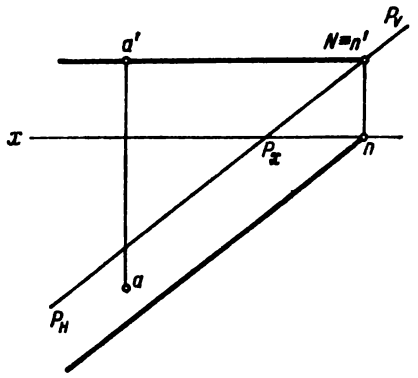


Fig. 87

The orthographic views given in Figs. 81-83 show the projections of the first three principal lines. The projections of these lines are characterized in the following table.

The characteristic feature of the steepest line is the perpendicularity of its horizontal projection to the horizontal trace of the plane.

Indeed, we consider three lines: MN, mn and P_H that intersect in the point M (see Fig. 84). The first is inclined to the H plane, the second is the projection of this oblique line, and the third is a perpendicular to the oblique line and is drawn in the H plane. But if MN is perpendicular to P_H, then by the theorem of three perpendiculars it may be stated that mn is perpendicular to P_H. Thus, the steepest line and its horizontal projection are perpendicular to P_H. They therefore form a linear angle which is used to measure the dihedral angle formed by the P and H planes.

Orthographically, the construction of a steepest line is begun from the horizontal projection, which is drawn perpendicular to the horizontal trace of the plane or perpendicular to the horizontal projection of any horizontal line of the given plane. The vertical projection of the steepest line may be determined by the vertical projections m' and n' of the traces of this line (see Fig. 85).

The principal lines of a plane are conveniently used to solve problems on the relative positions of a point and a plane. In Fig. 87 we have the plane P and the projections a and a' of a point A. We have to establish whether this point lies in the given plane. In the plane draw a horizontal line at the same level as the point A ($z_N = z_A$). The vertical projection of the horizontal line will pass through a' parallel to the x-axis. The vertical trace N of this line will be located in P_V, while the horizontal projection of the point N will lie on the x-axis. A second projection of the horizontal line will pass through n parallel to the trace P_H. The horizontal projection a of point A is found to be external to the like projection of the line. Hence, point A does not lie in the plane P.

Sec. 20. The Relative Positions of Two Planes

Two planes in space may be either mutually parallel, coinciding in the special case, or intersecting.

Mutually perpendicular planes are a special case of intersecting planes and will be considered below.

1. *Parallel planes.* Two planes are parallel if two intersecting lines of one plane are parallel to two intersecting lines of the other.

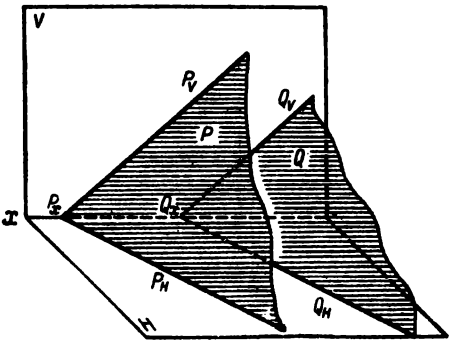


Fig. 88

We know that two parallel planes intersect a third along parallel lines. Thus, like traces of parallel planes are parallel (Fig. 88). Also parallel in this case will be the like projections of the corresponding principal lines.

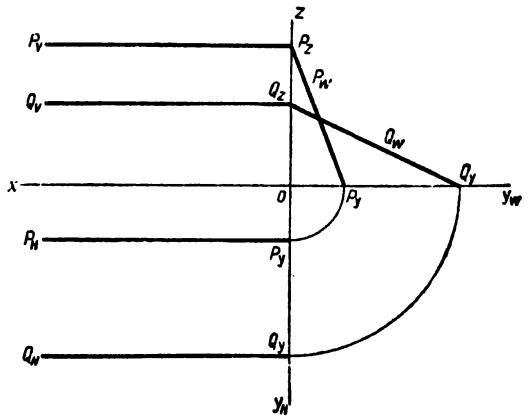


Fig. 89

To judge about the parallelism of profile projecting planes it is necessary to construct their profile traces. This is done in Fig. 89. The relative positions of P_W and Q_W indicate that the given planes intersect.

In a variety of problems it is often necessary to pass a plane Q through a given point A, the plane being parallel to a given plane P.

If P is determined by two intersecting lines A_1B_1 and A_1C_1, the required plane will be constructed by means of the lines AB and AC, which are respectively parallel to A_1B_1 and A_1C_1 and pass through the given point A (Fig. 90).

Now, in the case when the plane P is given by traces, it is sufficient to pass through A only one of the principal lines of the required plane, for example, a horizontal line (Fig. 91). Its vertical projection is

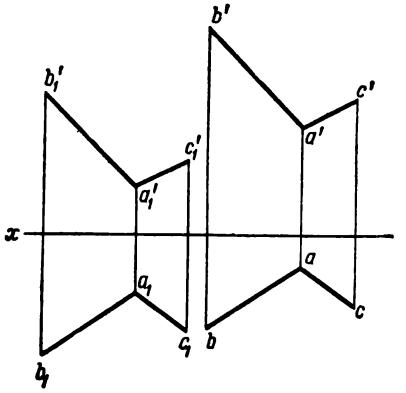


Fig. 90

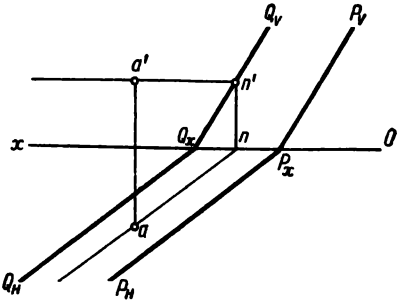


Fig. 91

parallel to the x-axis, while its horizontal projection is parallel to the trace P_H. The vertical trace of the desired plane will pass through the like trace of the horizontal line and will be parallel to P_V. Finally, a second trace of the desired plane is passed through the vanishing point Q_x and parallel to P_H.

2. *Intersecting planes.* The line of intersection of two planes is a straight line, for the construction of which it is sufficient to determine two points that are common to the two planes, or one point and the direction of the line of intersection of the planes.

Let us first consider a particular case of intersecting planes when one of them is parallel to the plane of projection. We shall demonstrate that a plane T, parallel to H, intersects any plane P along a horizontal line (Fig. 92). We know that two parallel planes (T and H) are intersected by a third (P) along parallel lines. Consequently, the desired line along which the plane P intersects T must be parallel to the line of intersection of P and H, which is the trace P_H. But a line in P, parallel to P_H, is a horizontal line. Now point N through which this line passes is determined by the intersection of the vertical traces of the given planes.

Orthographically, the problem is solved in Fig. 93. The vertical projection of the required line coincides with the trace T_V, and the horizontal projection is parallel to P_H.

With similar arguments it may be proven that the plane U, which is parallel to the V plane, will intersect any plane along the vertical principal line (Fig. 94). In this case, the horizontal projection of the line of intersection of the planes will coincide with U_H, while the vertical projection is parallel to P_V.

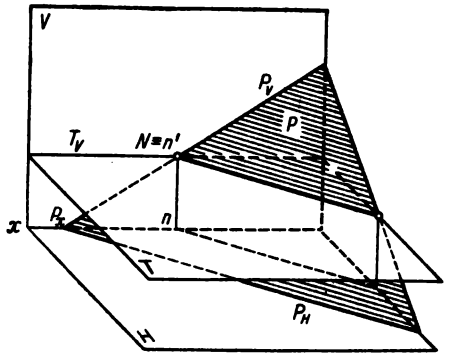


Fig. 92

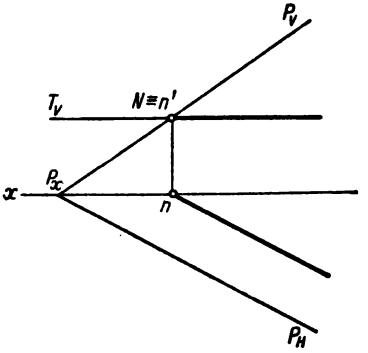


Fig. 93

Using this particular case of intersecting planes, let us consider the general case. Let there be given in space two oblique planes P and Q, the line of intersection of which it is necessary to construct (Fig. 95).

The given planes are intersected by an auxiliary plane T_1, which it is advisable to orient parallel to H or V. In our case, the plane T_1, which is oriented parallel to H, will intersect each of the given planes P and Q along the horizontal lines AB and CD, which determine the point I (common to the given planes P and Q and, hence, belonging to the line of their intersection).

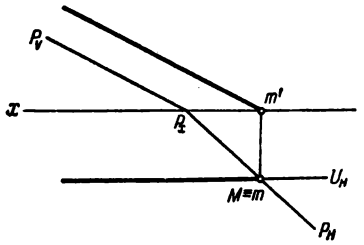


Fig. 94

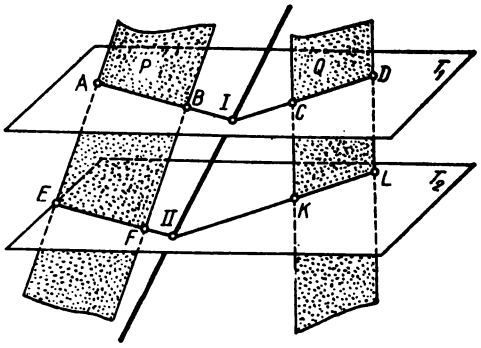


Fig. 95

Taking a second auxiliary plane T_2, which is likewise parallel to H, we obtain yet another point II, which is common to the planes

P and Q. This point is determined by the intersection of the horizontal lines EF and KL, along which the auxiliary plane T_2 intersects each of the given planes.

The foregoing method is used for orthographic construction of the projections of the lines of intersection of the two planes, the first of which is given by two parallel lines, and the second by three points (Fig. 96). The auxiliary plane T_1 yields point I as a point in which the horizontal lines EF and NN_1 intersect. In exactly the same way, plane T_2 yields the second point II.

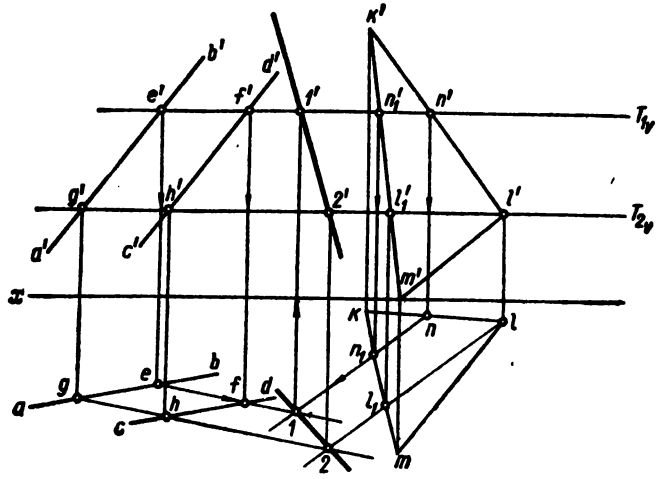


Fig. 96

The figures 97 to 102 illustrate the construction of the lines of intersection of two planes given by traces.

We shall note the peculiarities of each one.

In the first case (Figs. 97 and 98), the desired line is constructed with the aid of points in which like traces of the given planes intersect. These points, M and N, are the traces of the line, and since each of them at the same time lies on the traces of both planes, the line MN belongs to both planes, which means that it is the line of their intersection.

Fig. 99 illustrates a case when the point of intersection N of vertical traces of the given planes lies on the lower half-plane of V. The remaining constructions are similar to those executed in Fig. 98.

The peculiar feature of the next case (Fig. 100) is that the vertical traces of the given planes do not intersect within the limits of the drawing. One of the points M of the desired line is determined by

intersection of the horizontal traces of the planes; to find the other we have to resort to an auxiliary horizontal plane T. This plane cuts each of the given planes along horizontal principal lines.

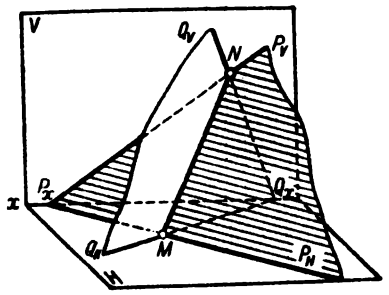


Fig. 97

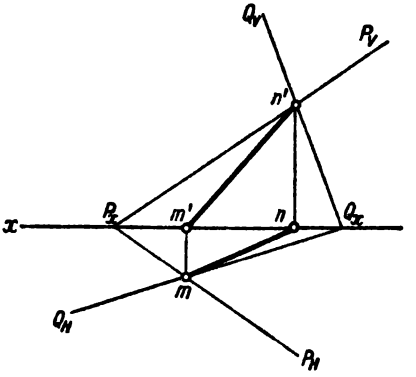


Fig. 98

Intersection of the latter is what determines the second point N of the required line. In this case, if the second pair of traces likewise fails to intersect within the limits of the drawing, it is necessary to take another auxiliary plane parallel to H or V.

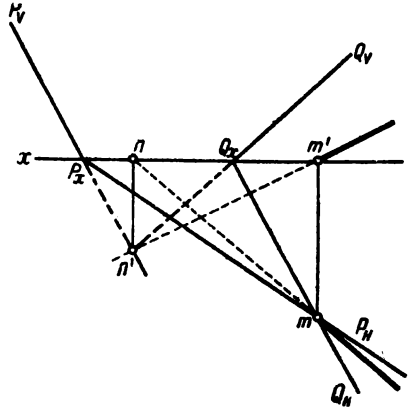


Fig. 99

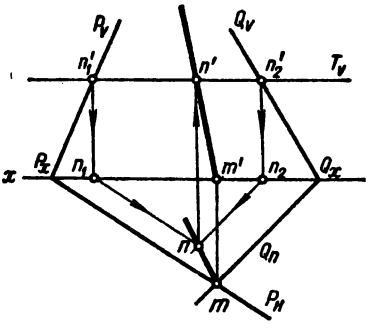


Fig. 100

In the next example, an auxiliary plane U, parallel to V, is used to find the second point, which is common to the two planes P and Q; these planes intersect the x-axis in the single point I (Fig. 101).

When one pair of plane traces are parallel lines (Fig. 102), the line of intersection of the planes is a straight line parallel to these traces.*

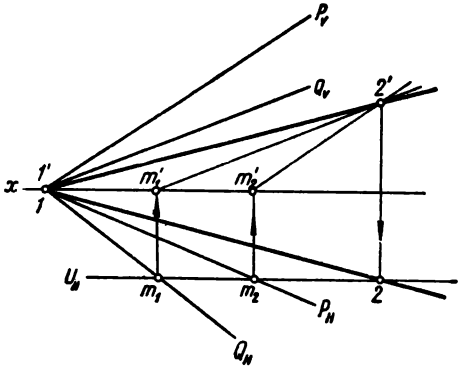


Fig. 101

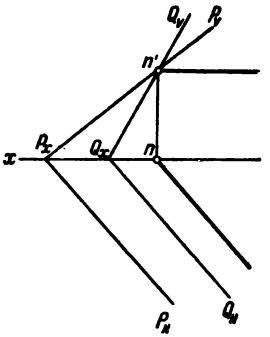


Fig. 102

In our case, two planes with parallel horizontal traces intersect along a common horizontal line which will pass through point N determined by the intersection of P_V and Q_V.

Sec. 21. The Relative Positions of a Straight Line and a Plane

There are three possible cases for the relative positions of line and plane:

(a) the line belongs to the plane,

(b) the line is parallel to the plane, and

(c) the line intersects the plane.

The first case was discussed in Sec. 18, in which we examined one of the basic graphical operations: the drawing of straight lines in a plane. The criterion here is the familiar property of a plane: if a straight line joins two points in a given plane, that line lies entirely in the plane.

Obviously, if a line does not have two common points with the plane, it is either parallel to the plane or cuts the plane. For the sake of definiteness in Fig. 103 we pass an auxiliary plane Q through the line AB and establish the relative positions of the two lines AB and MN, the latter of which is the line of intersection of the auxiliary

* This statement is based on the following theorem from solid geometry: two planes that pass through two given parallel lines and are not parallel to one another intersect on a straight line that is parallel to the given straight lines.

plane Q and the given plane P. To each of the three possible cases of relative positions of these lines there will be corresponding configurations of line and plane.

Thus, if both lines coincide, then AB lies in the plane P, parallelism of the lines will indicate parallelism of the line and the plane, and, finally, intersection of the lines will correspond to the case of AB cutting plane P. These two latter cases require a more detailed study.

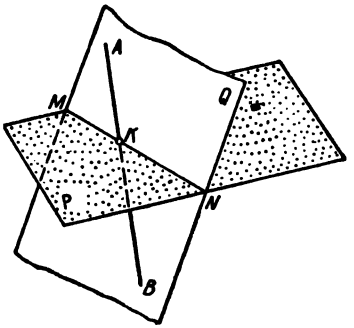


Fig. 103

Sec. 22. A Line Parallel to a Plane

When deciding on the parallelism of a line and a plane, it is necessary to invoke the familiar proposition of solid geometry: *a line is parallel to a plane if it is parallel to one of the lines lying in that plane.*

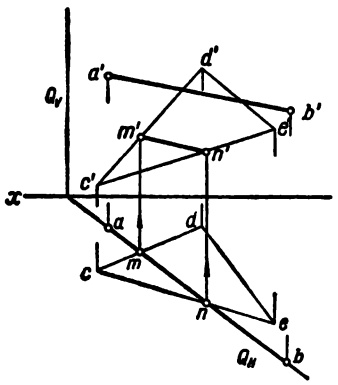


Fig. 104

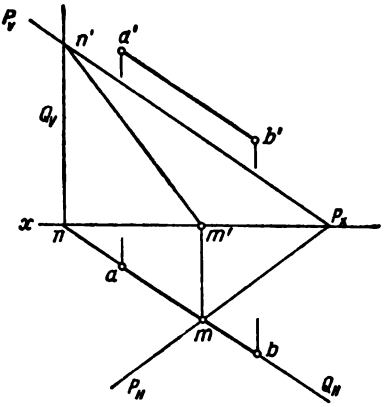


Fig. 105

Following the procedure explained in the preceding section, we evaluate the relative positions of AB and a plane (see Figs. 104 and 105). In Fig. 104 the plane is given by three points C, D, E, and in Fig. 105 it is given by the traces P_V and P_H.

To start with, in both cases pass an auxiliary plane Q through AB. For this purpose it is extremely convenient to make use of one of the projecting planes. In the above cases we pass through AB the horizontal projecting plane Q, whose horizontal trace merges with the

like projection of the line. We then construct the projections mn and $m'n'$ of the line of intersection of the planes, a comparison of which with the projections of the given line shows that AB is parallel to the plane of the triangle CDE (Fig. 104) and is not parallel to the plane P given by the traces (Fig. 105).

This latter conclusion could have been drawn even without the foregoing construction. Indeed, the vertical projection of the given line is parallel to P_V. We may therefore presume that the line AB is parallel to all vertical lines of the plane. But since the horizontal projection AB is not parallel to the x-axis, AB is not parallel to the plane P.

Sec. 23. A Line Cutting a Plane

The problem of this section is one of the basic problems of descriptive geometry. Further successful study of this course will depend largely on how well the student grasps the essentials of this problem.

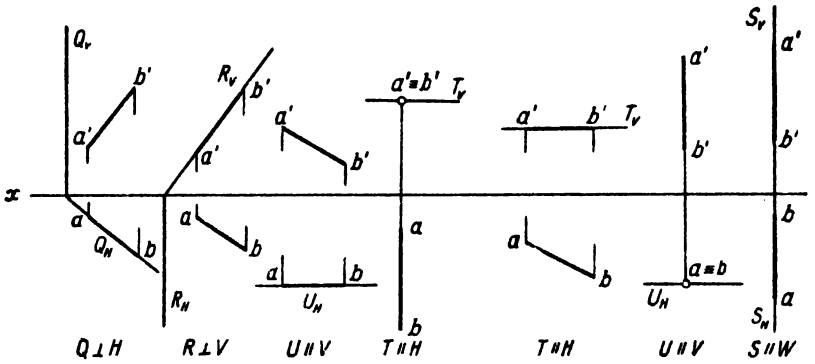


Fig. 106

It will suffice to enumerate a few of the problems that ultimately reduce to determining the point of intersection of a line and a plane: intersection of a line with a polyhedron, intersection of a polyhedron, cone, cylinder and any ruled surface with a plane, the intersection of two polyhedrons.

Following the procedure of Sec. 21, the solution of problems on the intersection of a line and a plane involve distinguishing three stages:

(1) construction of an auxiliary plane Q which is passed through the line AB;

(2) construction of the line of intersection of an auxiliary plane Q and a given plane P: the line MN;

(3) determining the desired point K as the point of intersection of two lines: the given line AB and the constructed line MN (see Fig. 103).

In Sec. 22 we pointed out that one of the projecting planes might be used for the auxiliary plane. Fig. 106 shows in what ways it is advisable to pass projecting planes through lines in a variety of space positions.

We consider the solution of some problems. Figs. 107 and 108 depict AB cutting the plane of the triangle CDE in the first case and the plane P (given by traces) in the second.

In both cases the point of intersection K is found by means of the horizontal projecting plane Q which intersects the given planes along

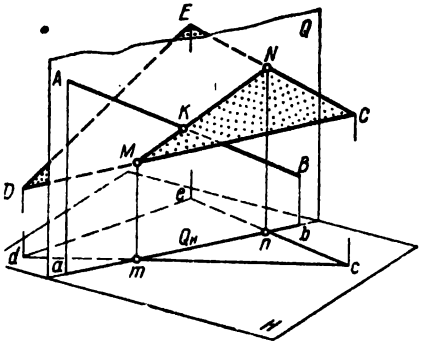


Fig. 107

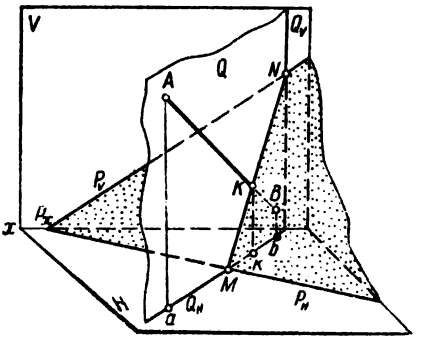


Fig. 108

the lines MN. In Fig. 107 the points M and N are the points of intersection of the sides of the triangle CD and CE with the plane Q. Since these points lie in the horizontal projecting plane, their horizontal projections lie at the intersection of cd and ce with Q_H. MN in Fig. 108 is determined by the points at which like traces of the planes P and Q intersect.

By constructing the line MN in each of the two examples, we find its point of intersection with the given line AB. This is the required point K.

The sequence for solving both examples orthographically is the same (see Figs. 109 and 110). In executing orthographic representations, pay special attention in the last stage of the solution where the projections of the desired point are determined.

Bear in mind that if the horizontal projecting plane is taken for the auxiliary plane, then the first one (of the two) to be determined will be the vertical projection of the desired point (Figs. 109, 110, 111). But when applying the vertical projecting plane, we first find the horizon-

tal projection k and then k' (Fig. 112). The orthographic solution culminates in determining the visible portions of the projections of the given line. The visibility of the projections of AB relative to the plane of the triangle CDE (see Fig. 109) is established with the help of special rays that are mentally passed through the point of intersection of the projections of the given line and the sides of the triangle as described in Sec. 12, which is devoted to the relative positions of two lines.

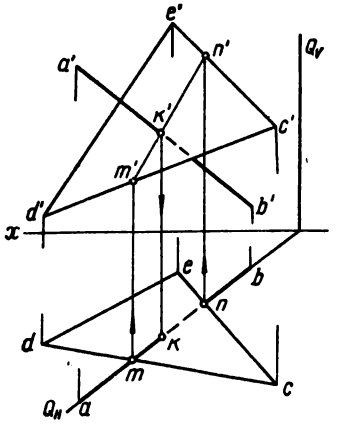


Fig. 109

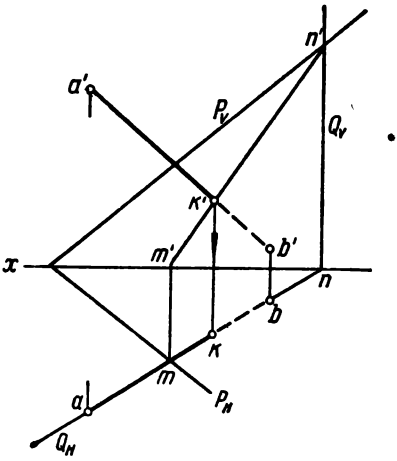


Fig. 110

When deciding the question of the visibility of a line in the first octant relative to a plane given by traces, remember that the portion of the projection that intersects the plane trace will always be visible. Point K (it is assumed that K also lies in the first octant) will separate the visible part of the line from the hidden part.

Fig. 112 shows how to determine visibility of a line when its projections are parallel to like traces of the plane. Here, to decide the question of the visibility of the vertical projection of the line, we take, on the line, an arbitrary point to the right or left of k' (for example, b') and draw through it a ray perpendicular to the V plane. With the aid of the very same vertical projecting plane Q, which passes both through the line AB and through the auxiliary ray, we find the point of intersection (I) of this ray with the P plane. The positions of the horizontal projections indicate that point B is farther away from the viewer than point I, hence point B is hidden by plane P.

To judge the visibility of the horizontal projection of a line one usually resorts to a similar construction, the only difference being that the auxiliary ray is oriented perpendicular to the H plane.

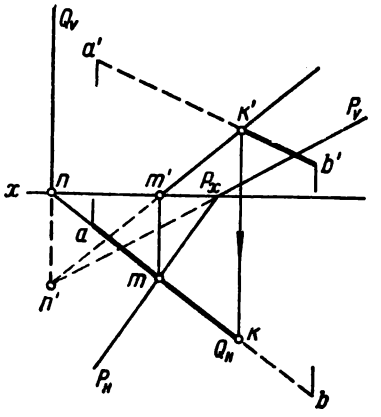


Fig. 111

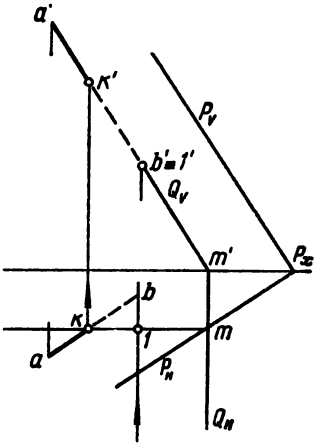


Fig. 112

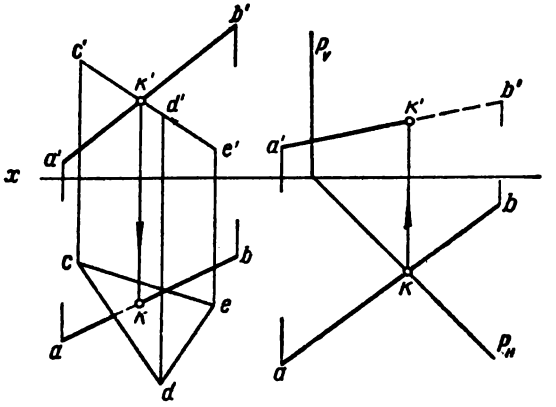


Fig. 113

Fig. 113 illustrates cases of the intersection of a line with a projecting plane. Since the desired points lie in projecting planes, one of the projections of each of them must lie on the respective trace. Thus, in the first example, k' is determined by the intersection of vertical

projections of the line and the triangle CDE, and in the second, the horizontal projection k of point K is found by the intersection of ab with P_H.

Sec. 24. A Line Perpendicular to a Plane

We prove the following theorem on a perpendicular to a plane. *If a line is perpendicular to a plane, then the projections of the line are perpendicular to like traces of the plane* (and also to the respective projections of horizontal and vertical lines).

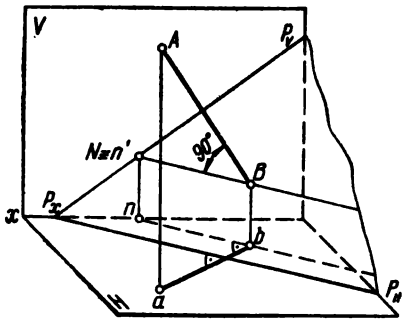


Fig. 114

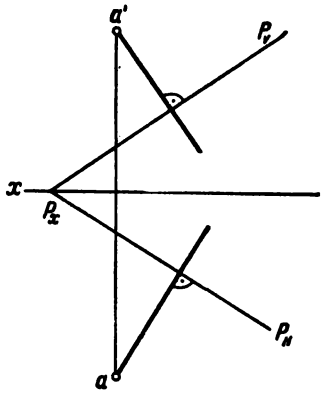


Fig. 115

In Fig. 114 we have a line AB perpendicular to the plane P. Let this line intersect P in point B. Draw the horizontal principal line BN through point B in plane P. Then, by hypothesis, AB is perpendicular to BN, and, invoking the theorem on the projection of a right angle, we may state that ab is perpendicular to bn. Now since P_H is parallel to bn, ab is perpendicular to P_H. Similarly, passing a vertical line through point B in plane P, we may prove that $a'b'$ is perpendicular to the vertical projection of the vertical line and to the trace P_V. The converse theorem also holds: *if the projections of a line are perpendicular to like traces of the plane, then this line is perpendicular to the plane.* Indeed, if the horizontal trace of plane P_H is perpendicular to the projection of the line, then it is perpendicular to the line itself as well. By virtue of this same theorem of three perpendiculars we may state that P_V is likewise perpendicular to this line. Hence, the line will be perpendicular to two lines, P_H and P_V, which lie in plane P, and so this line will be perpendicular to the given plane as well.

Fig. 115 shows the construction of projections of a perpendicular dropped from a given point A to plane P. The same problem is solved in Fig. 116 when a plane is given by the triangle BCD. Here the direction of the projections of the perpendicular was determined by the principal lines DE and DF of the plane of the triangle. Thus, the horizontal projection of the perpendicular is drawn at a right angle to the like projection of the horizontal line DE, the second projection of the perpendicular is situated at a right angle to the vertical projection of the vertical line DF.

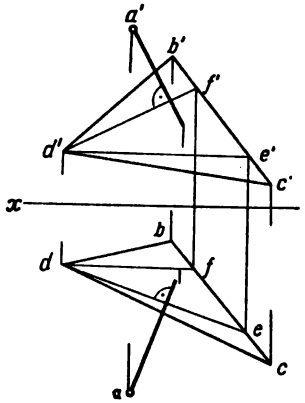


Fig. 116

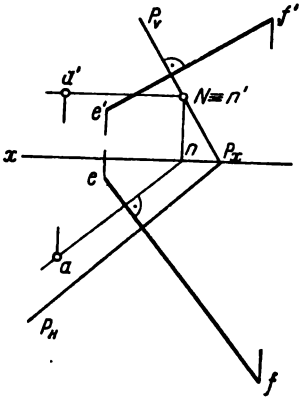


Fig. 117

One frequently has to solve the converse problem: to construct a plane P that passes through a given point A and is perpendicular to a given line EF (Fig. 117).

The traces of the desired plane P must be perpendicular to the like projections of the line EF. Since the directions of the traces are known, also known are the directions of the principal lines of this plane. And it is one of the principal lines (the horizontal line AN) that is drawn through point A. The horizontal projection of the horizontal line an passes through a and is perpendicular to ef. Now find its vertical trace N, through which P_V is constructed perpendicular to $e'f'$. The trace P_H is drawn at right angles to ef through the vanishing point P_x.

Sec. 25. Mutually Perpendicular Planes

From solid geometry we know that two planes are mutually perpendicular if one of them passes through a perpendicular to the other. Through a given point A we can pass an infinity of planes perpendicular to a given plane P. In space, these planes form a pencil of

planes, the axis of which is a perpendicular dropped from point A to plane P.

The orthographic view (Fig. 118) shows the construction of traces of one of the planes of this pencil. First, projections of the perpendicular to the plane P are drawn through the projections of point A at a right angle to the like traces of the plane. Then the traces of this perpendicular, M and N, are found. Draw Q_H through the horizontal trace M in an arbitrary direction. The second trace of the plane will be determined by the points N and Q_x.

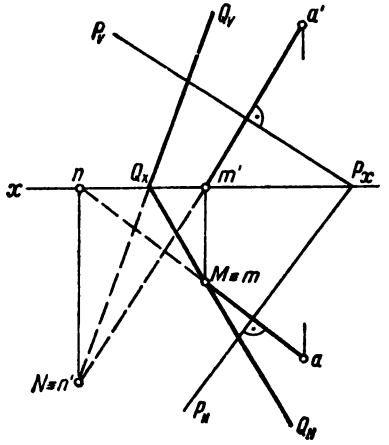


Fig. 118

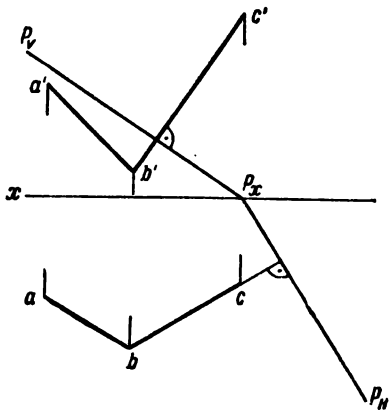


Fig. 119

We now pose the following problem: To pass a plane, perpendicular to a given plane P, through a given line AB (Fig. 119).

If the line AB is not perpendicular to the given plane P, then only one plane perpendicular to the given plane may be passed through this line. In Fig. 119 this plane is fully determined by two intersecting lines: the given line AB and the perpendicular BC dropped from an arbitrary point of the straight line AB on plane P.

Utilizing the test for mutually perpendicular planes, we can draw the conclusion (without determining the dihedral angle between them) either that the planes are perpendicular to one another or that they are not.

Given: two planes P and Q (Fig. 120). Required: to establish whether they are perpendicular or not. On one of them (plane Q) take an arbitrary point A, through which passes AB perpendicular to the other plane P. We now have to determine the position of AB relative to the plane Q. If AB lies in Q, then Q is perpendicular to P, other-

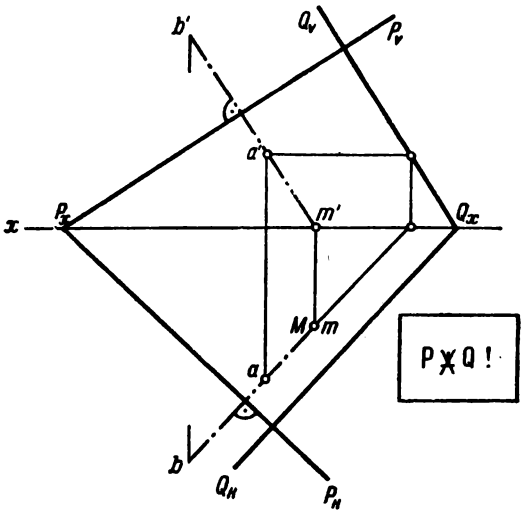


Fig. 120

wise the given planes are not perpendicular. The horizontal trace M of the line AB, obtained in Fig. 120, was found to be external to Q_H, which means that AB does not lie in plane Q.

Consequently, the planes P and Q are not perpendicular to one another.

CHAPTER IV

METHODS OF TRANSFORMING PROJECTIONS

Sec. 26. The Methods Characterized

The solution of metrical problems involving the determination of the true dimensions of figures and solids depicted orthographically may be extremely difficult if the given projections are not appropriately transformed.

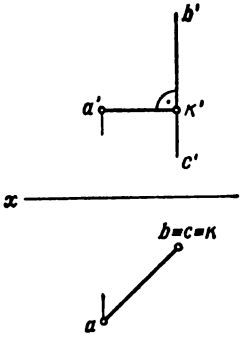


Fig. 121

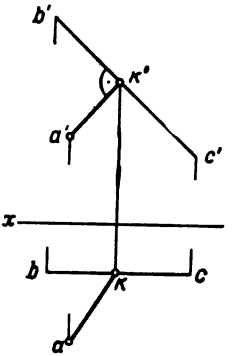


Fig. 122

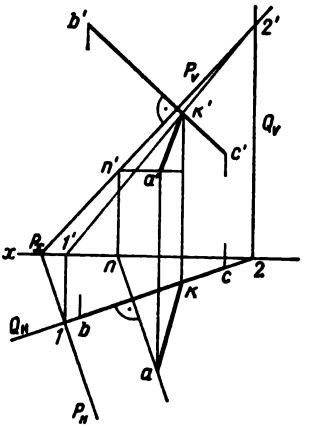


Fig. 123

Let us show how the complexity of solution of a specific problem depends on the given projections.

Figs. 121, 122, and 123 show projections of a point A and a line BC. In the first case BC is perpendicular to H, in the second BC is a vertical line, and in the last, it is an oblique line.

Required: to determine the distance of A to BC. The desired line-segment AK must be perpendicular to the line BC, and since in the first two cases BC is parallel to the V plane, a right angle between AK and BC is projected on this plane true size. But since in the first case BC is perpendicular to H, the line-segment AK (perpendicular to BC) will be parallel to H and its horizontal projection will determine the desired distance.

In Figs. 122 and 123 we were able to construct only the projections of the desired line-segment. In the latter case, this construction is rather unwieldy. Indeed, it was necessary to pass through point A a plane P perpendicular to BC, and then determine the point of intersection of the line BC with the constructed plane P. A comparison of these figures demonstrates that the difficulty of solution of the same problem depends very essentially upon the projections that are given. But the latter are ultimately determined by the position of the geometric image (point, line, figure, solid) relative to the planes of projection H and V.

Therefore, to pass from the two latter cases (see Figs. 122 and 123) to the first (see Fig. 121), retain the relative positions of the given point and line and change their positions relative to the planes of projection. One of two methods may be used for this purpose: revolution or replacement of planes of projection.

The revolution method consists in the fact that the position of a given geometric image relative to fixed planes of projection is changed by rotation about a certain axis.

Contrarily, the method of replacing the projection planes leaves the given geometric image fixed in position. The new planes of projection are established so that the resulting projections should ensure a rational solution of the problem at hand; note that each new system of projection planes must be an orthogonal system.

The methods of transformation of projections are not confined solely to metrical problems. Later on we shall show how these methods are used in the solution of so-called positional problems (problems involving the construction of the common elements of straight lines, planes and various geometric solids).

Sec. 27. Replacing Planes of Projection

The transformation of the projections of some solid executed by this method involves the transformation of the projections of points belonging to the solid. Let us therefore first consider the changes that the projections of a single point undergo when passing from one system of orthogonal planes of projection to another. Fig. 124 shows a point A given in a system of projection planes V/H. We replace one of them, say V, with another, which is also a vertical plane, V_1, and construct a new vertical projection of the point on this plane. Since the horizontal plane of projection H is common to the old and new systems, the z-coordinate of A remains unchanged. Therefore the distance from the new vertical projection to the new axis (O_1x_1) is equal to the distance from the replaced projection to the x-axis, or $a'_1a_{1x}=a'a_x$. Here, point a'_1 is determined as the foot of a perpendicular dropped from A to V_1. Now the horizontal projection a remains the same, while

the y-coordinate of point A will be different. This coordinate is determined by the distance from A to the plane V_1, which in the orthographic representation is equal to the distance from the projection a to the new axis O_1x_1; in this case the latter is drawn arbitrarily.

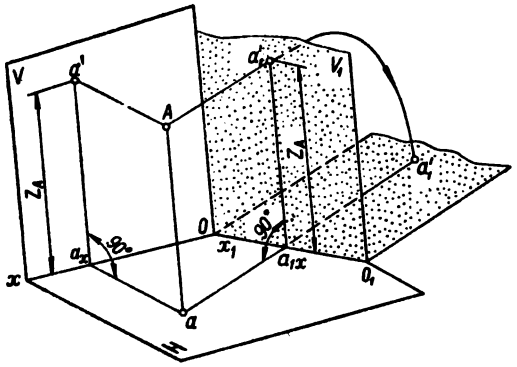


Fig. 124

To obtain the orthographic representation, rotate V_1 about O_1x_1 into H. Then the new vertical projection a'_1 of point A will coincide with H and will lie on the common perpendicular to the new axis O_1x_1 with the horizontal projection a that has remained unchanged (Fig. 125).

Similarly, we replace the horizontal plane of projection H with the plane H_1, which is also perpendicular to V (Fig. 126). This time there will be no change in the magnitude of the y-coordinate; the latter determines the distance from point B to the V plane, which is common to the two systems. Therefore the distance from the new "horizontal" projection of the point to the new axis O_1x_1 is equal to the distance from point b to the x-axis, or $b_1b_{1x}=bb_x$.

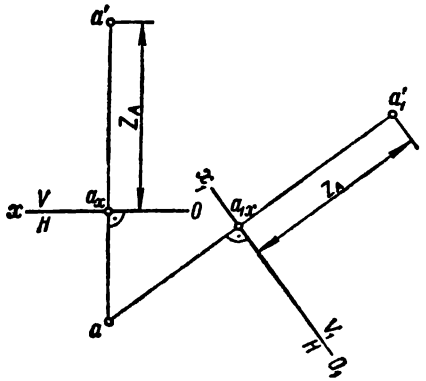


Fig. 125

When constructing a new projection of the point orthographically (Fig. 127), a perpendicular is dropped from b' to the new x_1-axis along which, from point b_{1x}, is laid off a line-segment $b_{1x}b_1$ equal to the coordinate y_B.

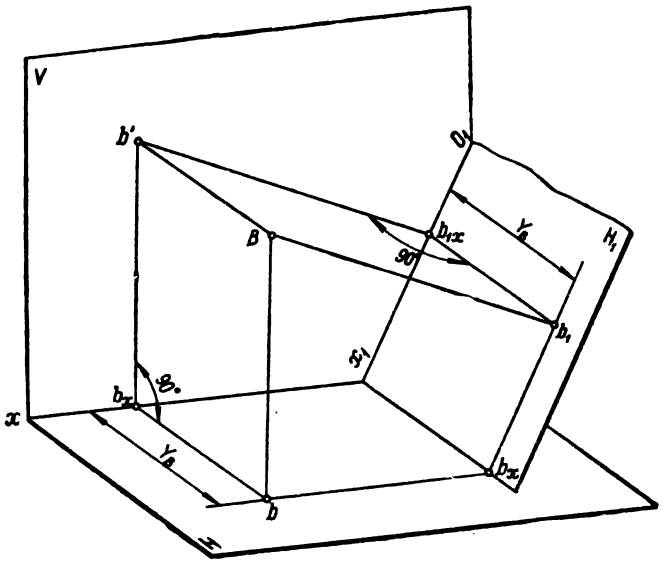


Fig. 126

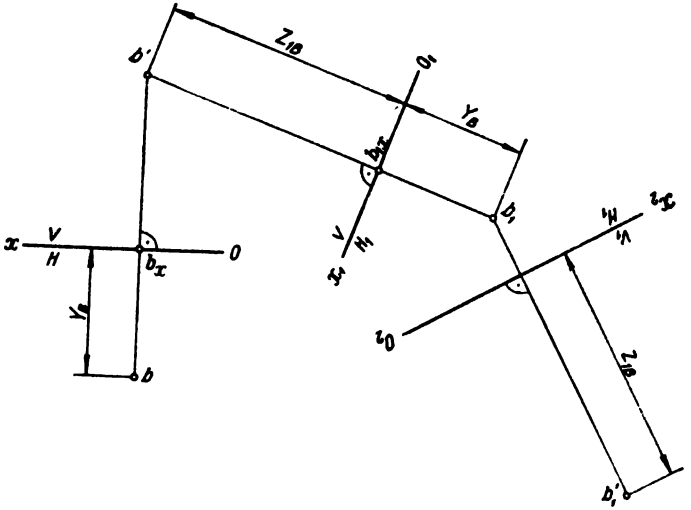


Fig. 127

Some transformations of projections require a double replacement of planes. Follow the rule given below during the successive transition from one system of planes of projection to another. The distance from the new projection of a point to the new axis must be equal to the distance from the projection (of the point) undergoing transformation (i.e., being replaced) to the previous axis.

In Fig. 127, during the first replacement, the previous axis of the x_1-axis was the x-axis, and therefore $b_1b_{1x}=bb_x$; in the transition from the V/H_1 system to the V_1/H_1 system the new axis became the x_2-axis, with respect to which the x_1-axis is the previous axis. We need only add that in all orthographic representations different (unlike) projections of points must be located on common perpendiculars to the appropriate axes.

Sec. 28. Solving Four Basic Problems by the Method of Replacing Projection Planes

All metrical and positional problems solved by method of replacement of projection planes may be reduced to one of the following four.

Problem 1. To transform the drawing so that an oblique line becomes parallel to one of the projection planes of the system.

Fig. 128 shows line AB which in the V/H system is an oblique line. To solve the problem, a new plane V_1 is taken that satisfies two conditions: V_1 is perpendicular to H and V_1 is parallel to AB. In

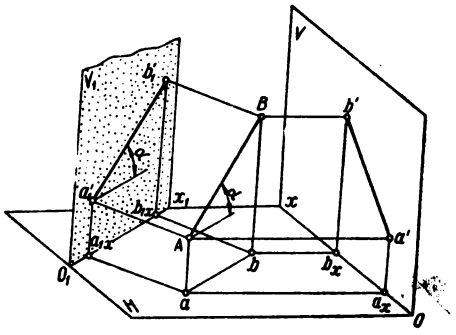


Fig. 128

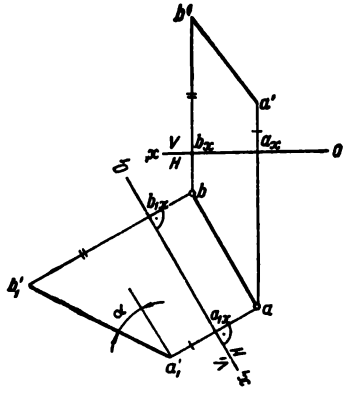


Fig. 129

the V_1/H system line AB has become a vertical line, and so O_1x_1 is parallel to ab. Now both segment AB and angle α project on plane V_1 true length and true size. Orthographically this problem is given in Fig. 129, where the x_1-axis is drawn parallel to ab, and a new vertical

projection of line-segment a_1b_1 is constructed in accord with the above rule. This problem may also be solved by replacing the horizontal plane of projection H with H_1 (Fig. 130). The new plane H_1 is perpendicular to V and parallel to AB, while the new axis (x_1-axis) is parallel to $a'b'$. Obviously, $a_1b_1=AB$ and angle β, which projection a_1b_1 makes with the x_1-axis is equal to the slope angle of AB to the V plane. This case is peculiar in one respect. Since the elements undergoing transformation—in our case, the horizontal projections of the

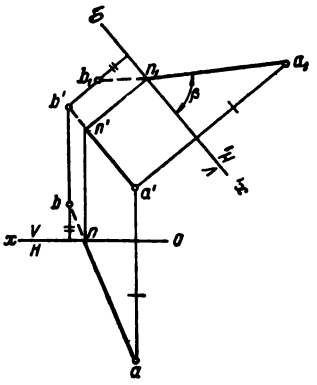


Fig. 130

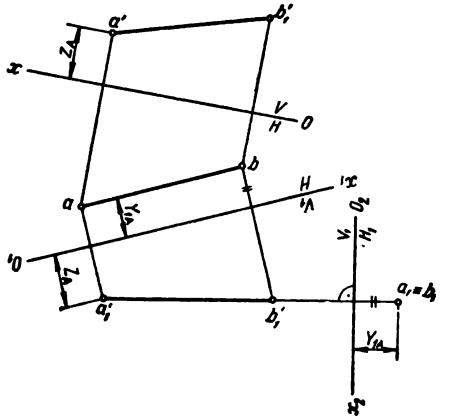


Fig. 131

extremities of the line-segment—are located on different sides of the x-axis, the new projections of these points, a_1 and b_1, must be on different sides of the new x_1-axis. This is due to the fact that $y_A>0$ and $y_B<0$.

Problem 2. To transform a drawing so that an oblique line is perpendicular to one of the projection planes of the new system. In other words, the line AB (Fig. 131) must become a projecting line in the new system.

Transformation of the projections of the oblique line AB into a point requires a double replacement of the planes because in the V/H system a plane perpendicular to AB will not be orthogonal either to V or to H.

When passing from the V/H system to the V_1/H system, plane V_1 is made perpendicular to H and parallel to line AB, which means that we solve the first problem that we have already considered.

In the second replacement, the new plane H_1 is drawn perpendicular to the line AB. This will ensure that the V_1/H_1 system is orthogonal The x_2-axis is constructed perpendicular to $a_1'b_1'$.

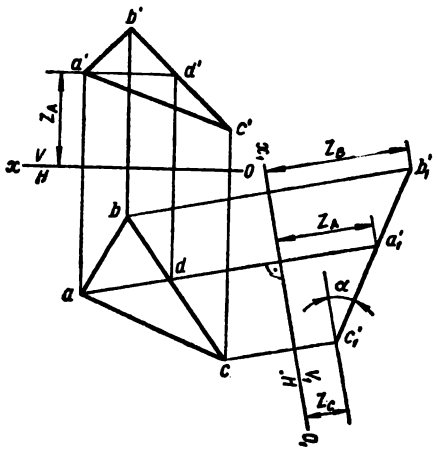


Fig. 132

Line AB will be a point on plane H_1. Thus, in the V_1/H_1 system, line AB has become a projecting line relative to the H_1 plane.

Problem 3. To transform a drawing so that an oblique plane becomes a projecting plane in the new system of planes of projection.

Let an oblique plane be given by three points A, B and C (Fig. 132). To solve this problem the new projection plane must be made perpendicular to the triangle ABC and one of the planes of projection. Which means that the new plane must be perpendicular to the line of intersection of the given plane and one of the projection planes. Here there is no need to construct the plane traces of the triangle since their directions may be found with the aid of principal lines.

That is why the first thing done in the given plane is to draw one of the principal lines, say, the horizontal line AD. This horizontal line is needed for orientation of the new plane of projection V_1.

By setting V_1 perpendicular to AD we ensure the fulfillment of two conditions: the new plane V_1 will be perpendicular both to H and to the plane of the triangle. The new x_1-axis, which must be perpendicular to the absent horizontal trace of the triangle, is drawn at a right angle to ad. Drawing, through the horizontal projections of the vertices of the triangle, lines perpendicular to the new axis, we lay off on these lines, from O_1x_1, line-segments equal to Z_A, Z_B and Z_C. In this way we get a new vertical projection of the triangle $a_1b_1c_1$ which is a straight line. It will be seen that the angle α (which the triangle makes with the H plane) is projected true size on plane V_1, which is perpendicular to the triangle and to H.

A similar transformation of the drawing has been executed in Fig. 133, where the H plane is replaced by plane H_1 perpendicular to V and to the triangle ABC. To achieve this, a vertical line AF was drawn in the plane of the triangle; H_1 is perpendicular to AF. The new axis O_1x_1 is chosen perpendicular to $a'f'$. The plane of the triangle relative to H_1 has become a projecting plane. The slope of the triangle to the vertical projection plane V is projected true size on plane H_1.

Now let plane P be given by traces (Fig. 134). A characteristic feature of a projecting plane in an orthographic representation is the perpendicularity of one of its traces to the x-axis. Therefore, if the

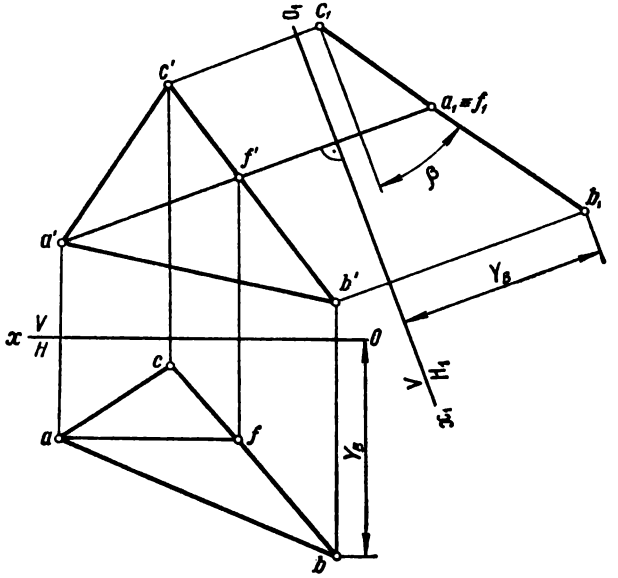


Fig. 133

problem is to transform plane P into a vertical projecting plane, the new axis, O_1x_1, is drawn at a right angle to P_H (in Fig. 134 O_1x_1 is perpendicular to P_H). In the new system of projection planes V_1/H, trace P_H will not change position.

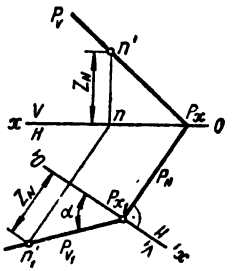


Fig. 134

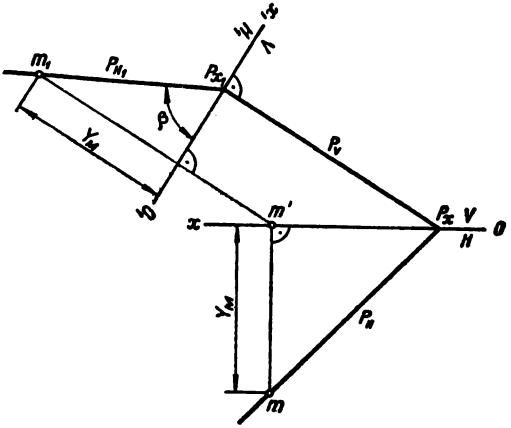


Fig. 135

There now remains the question of how to determine the new trace P_{V1}. In Fig. 134 the new position of the vertical trace was found with the aid of two points: the new vanishing point of traces P_{x1} and the new vertical projection n'_1 of point N contained in plane P. Trace P_{V1} must pass through n'_1 because the vertical projections of all points located in vertical projecting planes coincide with the like

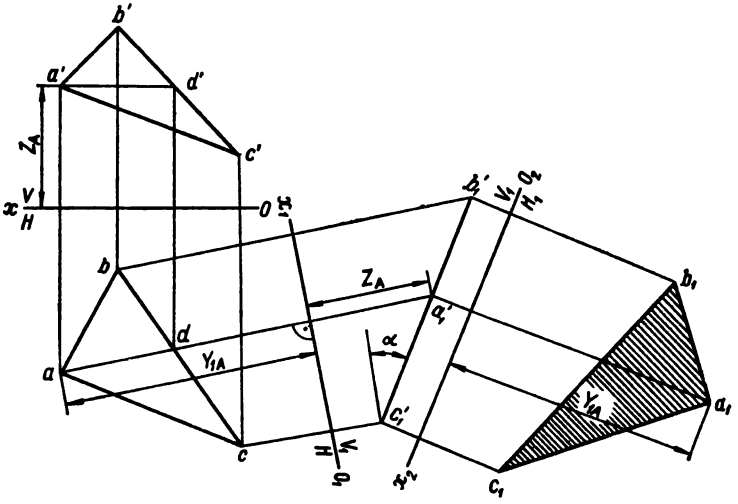


Fig. 136

trace. In Fig. 135, where plane H is replaced by plane H_1 (O_1x_1 is perpendicular to P_V), trace P_{H1} is determined by the points P_{x1} and m_1, the latter of which is the new "horizontal" projection of point M in the V/H_1 system.

Each of the executed transformations enabled us to determine the slope angles of plane P to the projection planes: α is the angle between P and H (see Fig. 134), β is the angle between P and V (Fig. 135).

Problem 4. To transform a drawing so that an oblique plane becomes parallel to one of the projection planes of the new system.

Given: a triangle ABC in an oblique plane (Fig. 136). Required: to construct a new orthogonal system of projection planes such that one of them is parallel to the triangle. Such a plane cannot be constructed in the V/H system. Indeed, a plane parallel to the triangle will not be perpendicular either to H or to V, which means that it does not form an orthogonal system with the projection planes.

The solution of this problem requires a double replacement of projection planes. The significance of the first replacement (V by V_1)

lies in the transformation of the plane of the triangle into a projecting plane. This process has been described above (see the solution of basic problem No. 3).

The second stage in the solution of the problem lies in a transition from the V_1/H system to the V_1/H_1 system. The new plane H_1 is established parallel to the triangle, which means that the new axis O_2x_2 in orthographic representation is drawn parallel to the line containing the points a_1', b_1' and c_1'. Then, as usual, we draw through these points perpendiculars to the new axis and on them lay off from O_2x_2 segments equal to Y_{1A}, Y_{1B}, Y_{1C} (Fig. 136 shows only the first). The projection $a_1b_1c_1$ thus constructed determines the true size of the triangle.

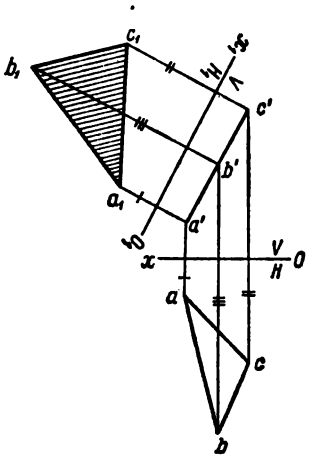


Fig. 137

But if the given plane is a projecting plane (Fig. 137), then the problem is solved by a single replacement of plares. In this case, plane H_1, which is parallel to the triangle ABC, forms with V the orthogonal system V/H_1. The new projection $a_1b_1c_1$ on H_1 determines the true size of the triangle.

Sec. 29. The Method of Revolution

1. *Revolution of a point.* Revolving about the axis II_1, point A describes a circle, the plane of which, P, is perpendicular to II_1 (Fig. 138). The centre of the circle Ω will lie at the point of intersection of the axis of revolution II_1 and the plane P (in which the point revolves), while the radius R will be defined as the distance of point

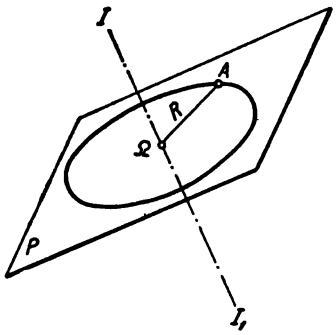


Fig. 138

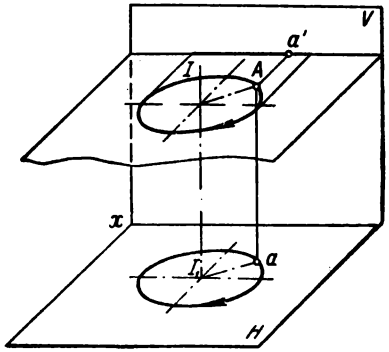


Fig. 139

A from the axis of revolution. If the projection plane is parallel to axis II_1, the projection of a revolving point on this plane will be a straight line perpendicular to the projection of axis II_1 on this same plane.

Let us first consider the revolution of the point about axes perpendicular to the planes of projection.

In Fig. 139 we have a pictorial view of point A revolving about axis II_1 which is perpendicular to H. In this case, the point will move in a circle in a plane parallel to H. This circle will be projected true

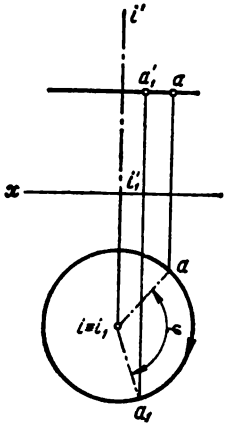


Fig. 140

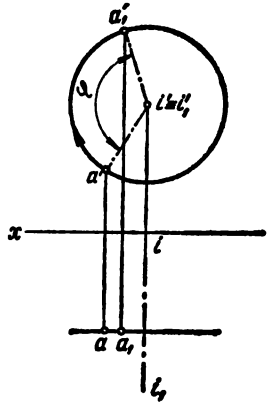


Fig. 141

size on H and in the form of a line-segment on the V plane, the line being parallel to the x-axis (Fig. 140).

Conversely, if the axis of revolution is perpendicular to the V plane (Fig. 141), then the horizontal projection of the point will move along a straight line parallel to the x-axis, and the vertical projection will move in a circle. In Figs. 140 and 141, A_1 is the new position of point A that it occupies after rotating through an angle φ.

Let us now consider an instance where a point is in revolution about an axis parallel to the H plane and not perpendicular to V or W (Fig. 142). The horizontal projection a of point A will in this case too be in motion along a straight line perpendicular to the projection ii_1 of the axis of revolution.

Indeed, axis II_1 is parallel to H and perpendicular to plane Q in which A revolves. Hence, planes Q and H are mutually perpendicular.

The circle that point A describes lies in the horizontal projecting plane Q and is projected on H in the form of a straight line coincident

with the trace Q_H. From the condition of the perpendicularity of a straight line and a plane, it follows that ii_1 is perpendicular to Q_H.

Thus, *if the axis of revolution II_1 is parallel to a certain plane P, the projection onto plane P of a point in revolution about the axis moves in a straight line. This line is perpendicular to the projection of the axis of revolution on the given plane P.*

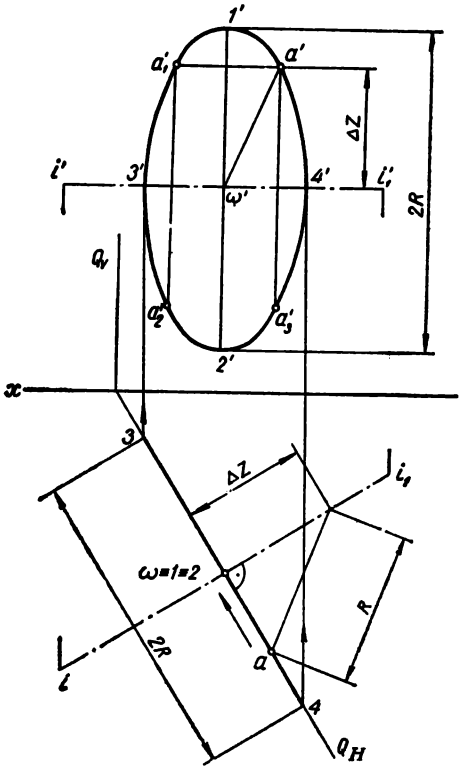


Fig. 142

Returning to our case, we see that the vertical projection a' of point A describes an ellipse.

The major axis of the ellipse will be equal to $2R$. We determine the radius of revolution R of point A as the hypotenuse of a right triangle, the other sides of which are line-segments ωa and Δz. The minor axis is constructed with the aid of points III and IV, which are the extremities of that diameter of the circle which is parallel to the H plane and is projected true size on H. As for points a'_1, a'_2 and a'_3, their

construction involved utilizing the properties of axial and central symmetry of an ellipse.

In similar manner we revolve a point about a vertical line.

2. *Revolution of a line.* We are given a line-segment AB and an axis of revolution II_1 perpendicular to the H plane (Fig. 143).

To construct a projection of a line-segment revolved clockwise about II_1 through an angle φ, it is sufficient to determine the new

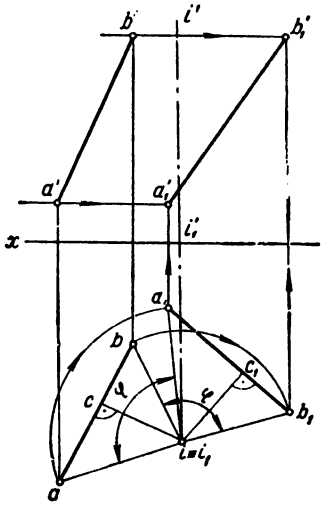


Fig. 143

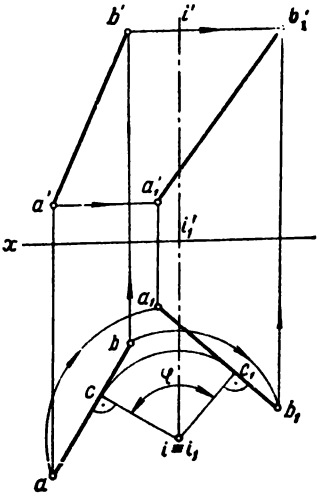


Fig. 144

position of two of its points, say, A and B. In the construction of new horizontal projections we fulfilled the condition: angle aia_1 equals angle bib_1.

The vertical projections of points A and B move along straight lines parallel to the x-axis. They will be determined by the intersection of these lines with the lines of recall drawn through points a_1 and b_1.

Noting that triangle aib = triangle a_1ib_1 (by two sides and the included angle), we conclude that their altitudes are equal: $ic = ic_1$. Taking advantage of this equality, the same revolution of line AB may be executed in the following manner (Fig. 144):

(1) from point $i \equiv i_1$ (the horizontal projection of the axis of revolution) drop the perpendicular ic on ab;

(2) revolve this perpendicular through the angle φ in the given sense into position ic_1;

(3) through point c_1 draw a straight line perpendicular to ic_1;

(4) at the intersection of the constructed straight line and arcs of radii *ia* and *ib* obtain points a_1 and b_1;

(5) construct the vertical projections a'_1 and b'_1 as in the preceding case (see the description pertaining to Fig. 143).

3. *Revolution of a plane.* To revolve a given plane P through an angle φ, it is sufficient to revolve two points of the plane through this angle. The new position of the plane will be determined by the revolved points and the fixed point of intersection of plane P with the axis of revolution. It is assumed here that the first two points do not lie on the same straight line as the third.

When the axis of revolution is parallel to the plane P, it is sufficient to revolve two points of P and then through one of them to draw a line parallel to the axis of revolution. Obviously, in this case these two points of plane P must not lie on a straight line parallel to the axis of revolution.

To determine the new position of the revolved plane, we can take a straight line (in place of two of its points) that does not intersect the axis of revolution and is not parallel to this axis.

Cases of the revolution of a plane about variously positioned straight lines will be considered below in the solution of basic problems.

Sec. 30. Solving Four Basic Problems by the Method of Revolution

Problem 1. To transform a drawing so that an oblique line, after a revolution through some angle, is parallel to one of the planes of projection.

If the line is parallel to the V or H plane, then one of its projections must be parallel to the x-axis. Hence, in solving the problem (to position the line AB parallel to V) we have to turn the horizontal projection of AB so that it is parallel to the x-axis.

To do this, the axis of revolution II_1 has to be chosen perpendicular to the H plane. In Figs. 145 and 146, axis II_1 is drawn through point A of the given line-segment. Point A will remain fixed as the line-segment is revolved. Any other point B of the segment and its horizontal projection will describe arcs of a circle. The angle of revolution of point B is determined by the condition of parallelism of the new projection ab_1 and the x-axis. As a result of such a turning, both segment AB and angle α (which AB makes with H) are projected true size on the V plane. By revolving the given line AB about an axis perpendicular to the V plane, it can be brought to a position parallel to the H plane (Fig. 147). In this case, the vertical projection of the segment should, upon being turned, become parallel to the x-axis. Both the line AB and angle β (which this line makes with the V plane) are projected true size on the H plane.

Fig. 148 illustrates the revolution of AB about the axis II_1, perpendicular to the plane W, and the determination of angle γ which AB makes with W.

As a result of this latter transformation, line AB becomes parallel to the V plane.

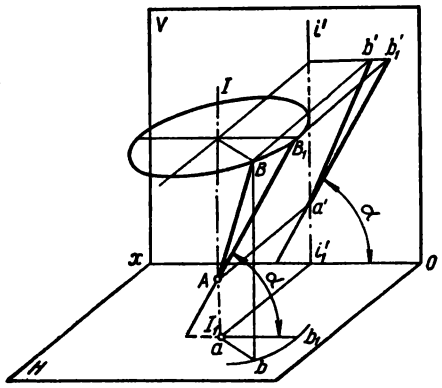


Fig. 145

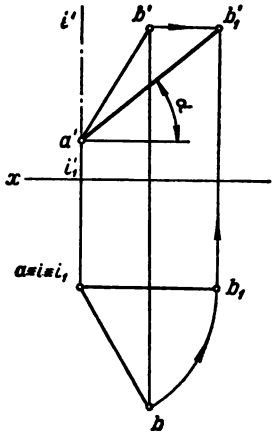


Fig. 146

Thus, a single turn about the projecting line brings the oblique line into a position parallel to one of the projection planes.

Problem 2. To transform a drawing so that an oblique line becomes a projecting line as a result of revolution.

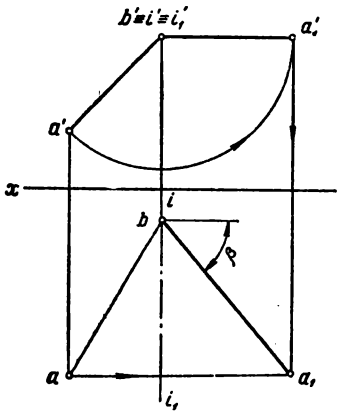


Fig. 147

This is attained by a double revolution of AB about two different axes (Fig. 149).*

The first revolution through the angle φ_1 is about the axis which passes through point A perpendicular to H. The line-segment is brought to a position parallel to the V plane. Corresponding to this position of the segment are the projections ab_1 and

* In Fig. 149 and most of the subsequent figures, the axes about which line-segments are revolved are not shown. But one should not forget the axes of revolution and should visualize them oriented relative to the projection planes.

$a'b'_1$. The second revolution through an angle φ_2 is executed about the axis perpendicular to the V plane and also passing through point A. As a result, the vertical projection of the segment $a'b'_2$ is perpendicular to the x-axis, the horizontal projection is converted to a point, $a \equiv b_2$, and the segment itself occupies a position perpendicular to the H plane.

Problem 3. To transform a drawing so that an oblique plane becomes a projecting plane after a revolution through some angle.

We consider the transformation of the plane of the triangle ABC into a vertical projecting plane (Fig. 150).

As we know, the characteristic feature of such a plane in an orthographic representation is the perpendicularity of the horizontal projection of its horizontal line to the x-axis.

That is why we first draw the horizontal line CD in the plane of the triangle ABC; CD is brought to position CD_1 (perpendicular to the V plane) by revolution through the angle φ_1 about axis II_1.

Since it intersects the axis of revolution, one revolved horizontal line does not determine the new position of the plane of the triangle. For this reason, the vertices A and B are also turned through the angle φ_1. The vertical projection of the triangle is converted into a straight line. With the x-axis it makes the angle α, which is equal to the slope angle of triangle ABC to the H plane.

Let us consider a case when the plane P is given by traces (Fig. 151).

Plane P intersects the axis of revolution (which is perpendicular to H) in point K. Plane P_1, which is turned through the angle φ, also passes through this point. We determine the position of this plane by the point K and one of the lines, the horizontal trace, for instance. In Fig. 151, plane P is transformed (by revolution) into the vertical projecting plane. After the revolution, its trace P_{1H} is perpendicular to the x-axis. And the distance from the axis of revolution to the trace remains unchanged, that is, $im = im_1$ Pass the vertical trace of the plane through the new vanishing point P_{1x} and k'. This transformation has enabled us to determine the slope angle of plane P to H (angle α).

The orthographic constructions involved in its transformation may be simplified if the axis of revolution is located on the V plane. Then this axis will intersect the P plane in point K, for the location of which no auxiliary constructions are required (Fig. 152).

Problem 4. To transform a drawing so that an oblique plane, through revolution, becomes parallel to one of the planes of projection.

We first give the solution when projecting lines serve as the axes of revolution. The first revolution of the triangle ABC was about the vertical axis which passes through vertex C (Fig. 153).

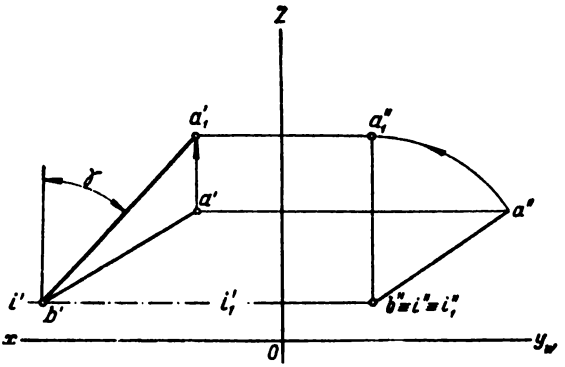
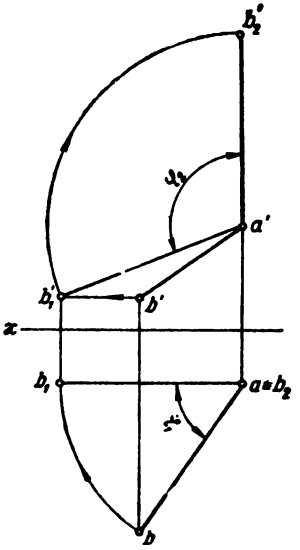


Fig. 148



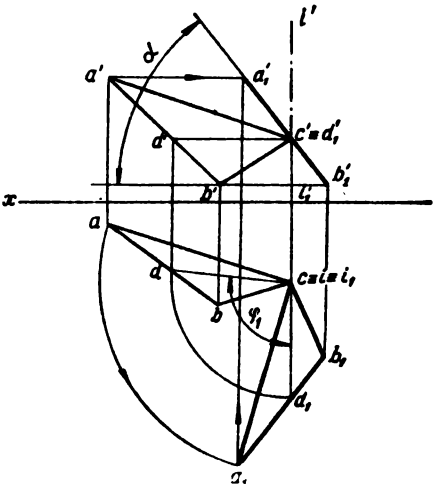


Fig. 150

As a result, the oblique plane becomes the vertical projecting plane; in other words, the first stage of the transformation is an exact repetition of the solution of Problem 3. After that, a second revolution

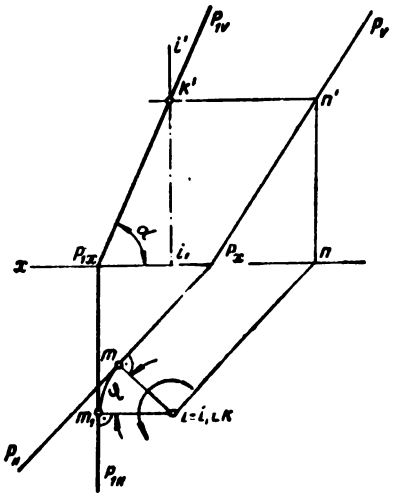


Fig. 151

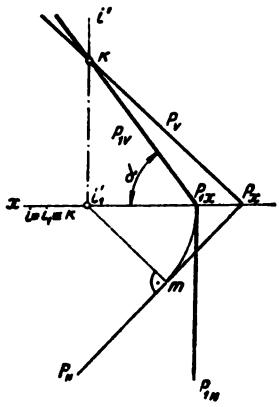


Fig. 152

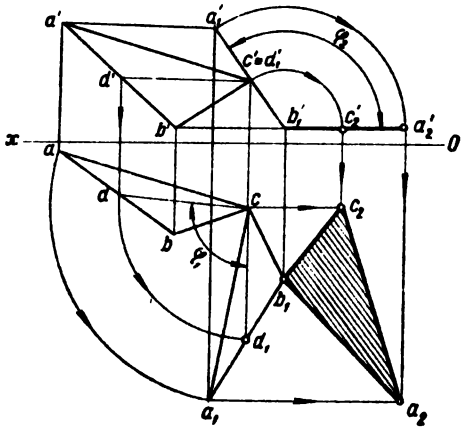


Fig. 153

may be performed through the angle φ_2 about an axis that passes through vertex B_1 perpendicular to the V plane. The vertical projections of all vertices of the triangle will be translated through concentric arcs drawn from point b_1' as from a centre, while the horizontal

projections will shift along lines parallel to the x-axis. After revolution through angle φ_2, the plane of the triangle becomes parallel to H. Hence, the shaded horizontal projection of the triangle determines its true size.

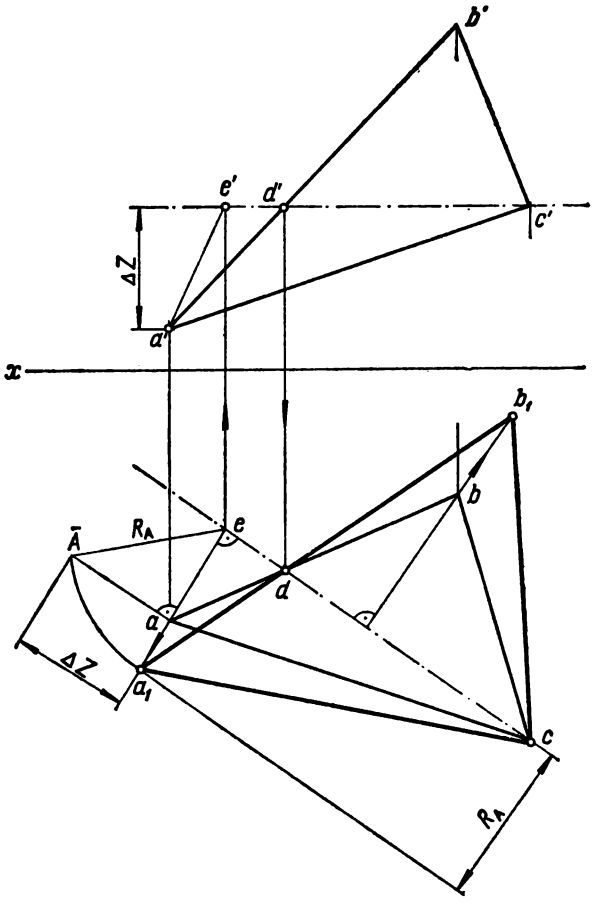


Fig. 154

We now demonstrate that this problem may be solved by a less cumbersome method: by revolution about a single axis.

If our aim is to position the triangle parallel to the H plane in a single revolution, then we should take for the axis of revolution such a line in the plane of the triangle as was parallel to H prior to the revolution, that is, one of its horizontal lines. In Fig. 154, line CD

is just such a horizontal line. Without repeating all the details given in Item 1 of the preceding section, where we considered the revolution of a point about a horizontal line, we note the principal feature of the construction: at the instant that the plane of the triangle is parallel to H, the horizontal projections of each of the moving vertices will be at a distance from the axis of revolution that is equal to the radius of revolution of the given point.

Subsequent operations follow in this sequence:

1. Draw straight lines perpendicular to cd along which the horizontal projections of the moving points are to move.
2. Construct the projections of the radius of revolution of one of them, for instance A. These will be the line-segments ae and $a'e'$.
3. From two projections we determine the true length of the radius of revolution R_A.

In Fig. 154 the radius R_A is defined as the hypotenuse $\overline{Ae}$ of the right triangle $\overline{A}ae$. Note that $\overline{A}a=z_E-z_A$.

4. Lay off segment R_A from point e on the straight line along which the horizontal projection of vertex A is in motion.
5. Draw a line through the point a_1 thus obtained and through the fixed point d to intersection with the line along which the horizontal projection of vertex B is in motion.
6. Joining the points a_1 and b_1 thus found and connecting them with the fixed vertex C, we determine the new horizontal projection of the triangle.

This projection yields the true size of triangle ABC.

The vertical projection of the triangle is thus transformed into a straight line that coincides with $c'd'$.

Sec. 31. The Coincidence Method

Coincidence is a special case of the method of revolution of a plane when one of its traces serves as the axis of revolution. As a result of such revolution, the given plane is brought into coincidence with one of the projection planes. In the latter plane we get a true size representation of the figures contained in the plane brought to coincidence. Essentially, this special case of the method of revolution does not contain anything fundamentally new. We therefore confine ourselves to a description of a few of its peculiarities.

Figs. 155 and 156 show constructions that result from coincidence of an oblique plane P with the H plane.

Here, the horizontal trace P_H of the plane is the axis of revolution. All points of this line remain fixed. Any other point of plane P, for instance A, will describe an arc of a circle in a plane perpendicular to the axis of revolution, and its horizontal projection a will move

along the line aM, which is likewise perpendicular to the axis of revolution (to the trace P_H). After coincidence with the H plane, point A must lie on line aM at a distance from P_H equal to the radius of revolution of point A. The length of radius AM may be defined as the hypotenuse of a right triangle with legs Aa and aM.

However, by making the H plane coincident with an arbitrary point A of the P plane, there is no need to determine the radius of revolution AM. It is advisable to do otherwise: first bring to coincidence

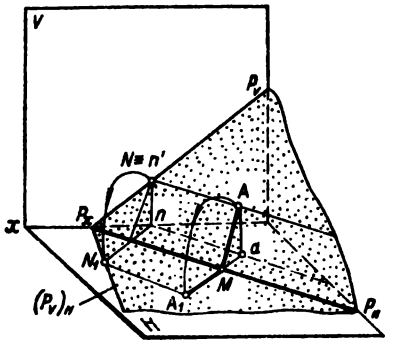


Fig. 155

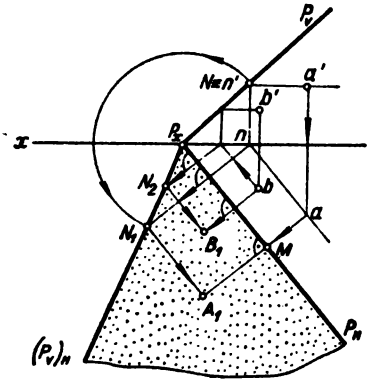


Fig. 156

with H one of the straight lines passing in the P plane through the given point A, and then determine the position of A on this line.

For this line it is common to take a line parallel to the axis of revolution. In the case under consideration (see Figs. 155 and 156), a horizontal line AN is drawn through point A in the P plane. When revolved about the trace P_H, the horizontal line AN will remain parallel to this trace. To construct a new position of the horizontal line, its vertical trace (point N) is brought to coincidence with H. The coincident position N_1 of point N must be on the line along which its horizontal projection n moves and at a distance NP_x from the fixed vanishing point P_x of the traces.

In the orthographic representation (see Fig. 156), a straight line nN_1, which is perpendicular to the axis of revolution (to the trace P_H), is drawn through the horizontal projection n of the vertical trace N of the horizontal line AN. The horizontal projection of point N moves along this straight line. Then from point P_x as a centre we describe an arc of radius P_xN to intersection with the line nN_1. We draw a horizontal line, brought to coincidence with H, through the point N_1 thus found; it is parallel to the axis of revolution.

The intersection of the horizontal line with the perpendicular to P_H, along which a moves, will determine the desired point A_1. In the case of coincidence of a number of points of plane P (this must sometimes be done to determine the true size of figures), construction of coincident vertical traces of auxiliary lines (horizontal lines) is somewhat simplified because these traces are located on a single straight line: the vertical trace (coincident with H) of the plane. Let us agree to designate as $(P_V)_H$ the vertical trace (coincident with H) of the plane P. Its position is determined by two points: the fixed point P_x and the constructed point N_1.

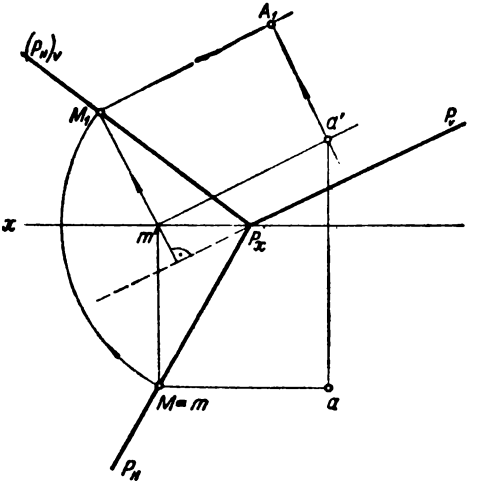


Fig. 157

Fig. 156 shows how $(P_V)_H$ is used to construct a coincident position of point B_1 contained in plane P.

Fig. 157 shows the coincidence of point A with plane V by means of revolution about P_V.

In this case, point A is "linked" with plane P by a vertical line. First of all, the horizontal trace of the vertical line is coincident with plane V. Its projection m' moves along a line perpendicular to the axis of revolution (to trace P_V). It is the intersection of this line with the arc of radius P_xM described from P_x that determines the horizontal trace (coincident with the V plane) of the vertical line. The vertical line coincident with V will pass parallel to P_V through point M_1. The intersection of the vertical line with the perpendicular to P_V, along which a' moves, yields the required point A_1. Joining M_1 and the fixed vanishing point P_x, we get $(P_H)_V$, which is the horizontal trace of the P plane; this trace is coincident with V.

Fig. 158 illustrates the use of the coincidence method for determining the true size of a plane figure. Two projections of triangle ABC enabled us to construct the horizontal trace P_H of the plane of the given triangle. This trace P_H is taken as the axis of revolution. The horizontal projections of all vertices of the triangle will move perpen-

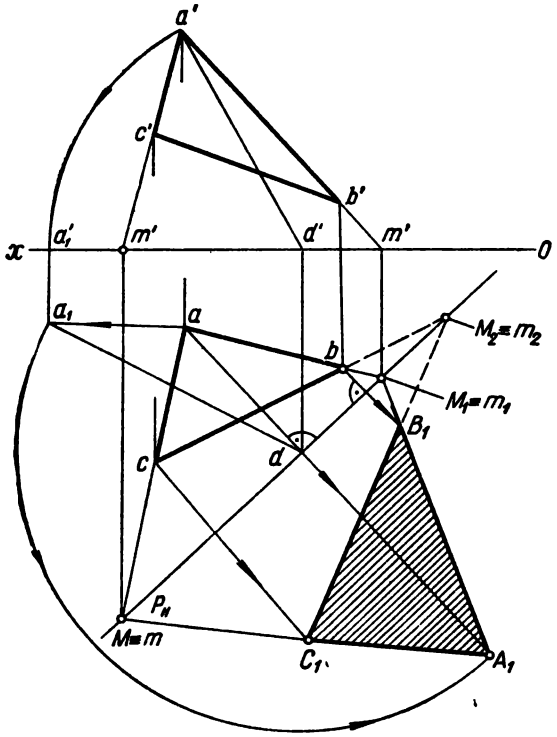


Fig. 158

dicular to the trace. The radii of revolution of each of them will determine the distances from P_H to the coincident position of the corresponding vertex. To construct the true-size triangle we need only determine the radius of revolution AD of only one of its vertices (A, for instance) which is done in Fig. 158 by the revolution method.

Naturally, we first had to construct the projections of this radius ad and $a'd'_1$. We take the segment da_1 thus found and lay it off from point d on the line along which a moves. This will yield the coincident position of vertex A_1.

The sides of the triangle (AC and AB), when extended, intersect P_H in points M and M_1, which do not alter their position upon revo-

lution. The sides will go through their extensions even after coincidence. Joining A_1 with M and M_1, we get lines that must contain the vertices B_1 and C_1. Their positions will be defined by the intersection of the lines A_1M and A_1M_1 with those perpendiculars to P_H along which the horizontal projections b and c move.

Other cases of the use of the method of coincidence in the solution of metrical problems will be given in Sec. 32.

Sec. 32. Applying Methods of Transformation of Projections to the Solution of Metrical Problems

1. Determining distances

(1) *The distance between two points.* The problem reduces to determining the true length of a segment connecting two given points. Its solution involves transforming the drawing to make the given segment parallel to one of the projection planes (see the first of the four basic problems considered above).

(2) *The distance from a point to a straight line.* In Sec. 26 it was shown that the complexity of solution is largely dependent on the projections that are given. Of the three cases indicated in Figs. 121, 122 and 123, the simplest is the first, when the line BC makes a right angle with H and is projected into a point on this plane.

But we can always arrive at this particular case of the position of line BC by utilizing the constructions given above in the solution of the second basic problem.

It will be recalled that the transformation of projections of an oblique straight line into a point requires a double replacement of planes, as is done in Fig. 159. The line BC has become a projecting line relative to plane H_1, and its projection on H_1 will be the point $b_1 \equiv c_1$. The given point A is also projected on this plane H_1. The distance between the new projection a_1 of point A and the new projection $b_1 \equiv c_1$ of line BC will be the desired one.

Fig. 159 also shows the reverse process of transforming the projections of line-segment AK from the system V_1/H_1 to the original system V/H. It will be noted that the projection of segment AK on plane V_1 is constructed parallel to the axis O_2x_2 because this segment isparallel to plane H_1.

(3) *The distance between two parallel straight lines.* In Fig. 160, the projections of two parallel oblique lines are transformed into points by a double replacement of the projection planes. The distance between them will be the desired distance. Indeed, in the second replacement of the projection planes, plane H_1 is located at a right angle to the given lines. Hence, a perpendicular dropped from any point

of one line onto the other will be parallel to plane H_1 and will be projected true length on it.

(4) *The distance between two skew lines* (Fig. 161). This distance is measured by the length of the perpendicular EF, which is common to the given lines. If one of them, AB, for example, is perpendicular to the H plane, then the common perpendicular EF (being a straight line making a right angle with AB) will be parallel to the H plane.

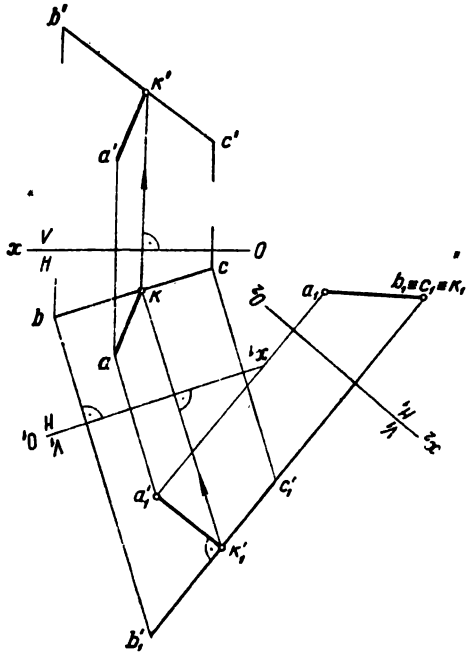


Fig. 159

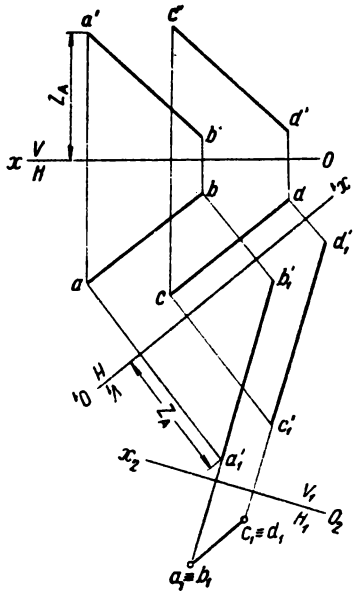


Fig. 160

But then the right angle between EF and the second of the skew lines CD, that is, angle EFD will be projected true size on plane H. It is for this particular case (where AB is perpendicular to H) that the solution of the problem is given in Fig. 162. A characteristic feature of the example shown in Fig. 163 lies in the fact that one of the skew lines lies parallel to the V plane. Here a single replacement of projecting planes will suffice to pass over to the special case, the solution of which is given in the preceding orthographic representation. The constructions in the V/H_1 system in Fig. 163 do not differ in any way from the constructions performed in Fig. 162.

In the general case, when neither of the skew lines is parallel to any one of the projection planes, the problem reduces to transforming

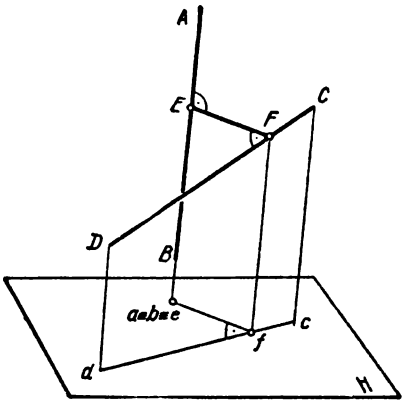


Fig. 161

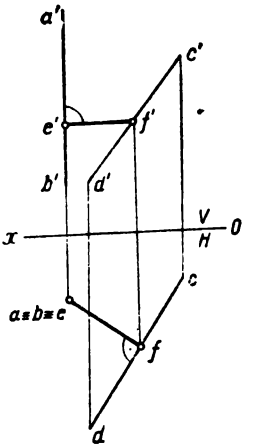


Fig. 162

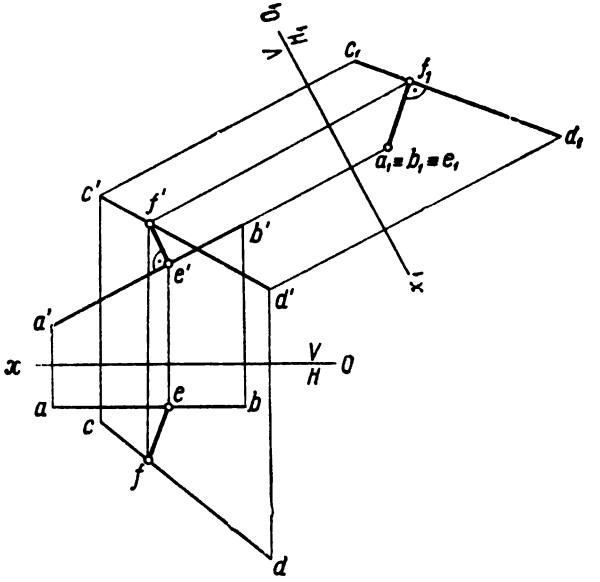


Fig. 163

the drawing so that the projection of one of the given lines becomes a point. This may be attained either by a double replacement of the planes, or by a double revolution of the system of skew lines (see second basic problem).

(5) *The distance from a point to a plane.* This distance is measured by the length of the perpendicular dropped from the given point A on the plane P (Fig. 164). First, draw projections ak and $a'k'$ of the

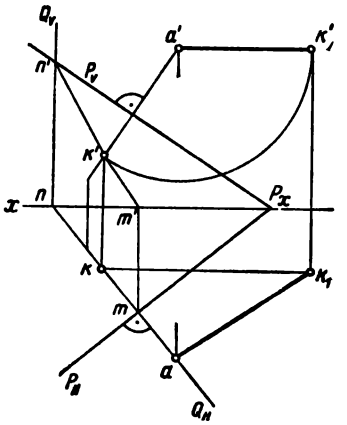


Fig. 164

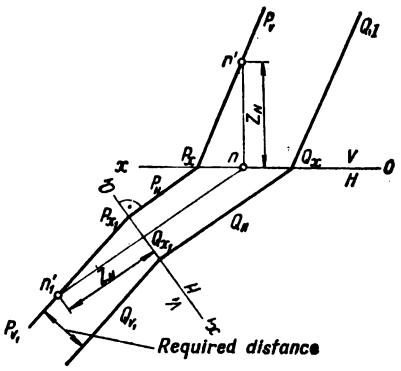


Fig. 165

perpendicular at right angles to the plane traces. Then, using an auxiliary plane Q, find the point of intersection K of the perpendicular and the plane P. The true length of the desired distance (segment ak_1) is determined by the method of revolution.

(6) *The distance between two parallel planes.* In this case, the distance is measured by the length of a perpendicular dropped from an arbitrary point of one plane onto the other. Thus, this problem reduces to the preceding one.

However, the most rational is the solution given in Fig. 165, where both planes are transformed into projecting planes by the method of replacing planes (see the third basic problem). The distance between the newly produced vertical traces of the given planes will be the required distance.

2. Determining angles

(1) *An angle between two intersecting lines.* If a plane defined by two intersecting lines AB and AC is positioned parallel to H, the angle BAC will be projected true magnitude on the H plane.

For this purpose (see Fig. 166) a horizontal line BC is drawn in the plane of the given lines and is taken as the axis of revolution.

The subsequent construction is similar to what has been described above (see the solution of the fourth basic problem by the revolution method). It will be noted that in Fig. 166 the centre of the circle which vertex A describes is labelled O.

(2) *An angle between two skew lines.* The measure of this angle is the angle between two intersecting lines that are parallel to the

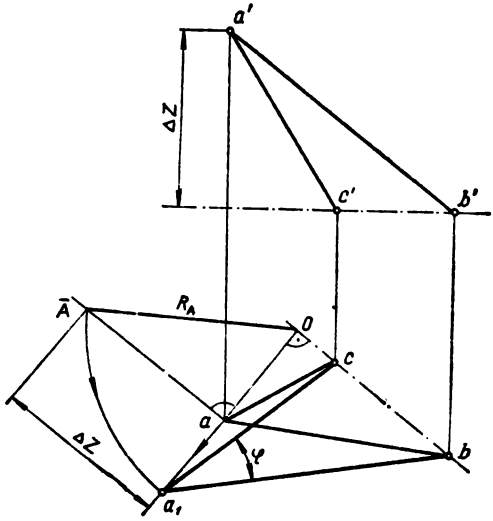


Fig. 166

given skew lines. Hence, in this case too the problem reduces to determining the true size of a triangle, and this may be done by any one of the known methods (see the solution of the fourth problem).

(3) *An angle between a line and a plane.* The angle between line AB and plane P (Fig. 167) is the acute angle φ which this line makes with its projection on the given plane. Construction of the projections of angle φ requires us to determine two points K and K_1, the former being the point of intersection of the given line with the P plane, and the latter, the foot of the perpendicular dropped from an arbitrary point A of the line on the same plane (P). With two lines thus intersecting in point K, we determine the true magnitude of the angle between them, as described in Item 1.

If the problem requires a determination of only the magnitude of the angle between the line and the plane without any depiction of its projections, then the solution is greatly simplified by dropping the construction of points K and K_1.

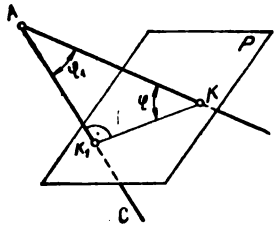


Fig. 167

Indeed, considering the right triangle AK_1K (see Fig. 167), we note that $\varphi=90°-\varphi_1$, where φ_1 is the angle made by the given line AB and the perpendicular to the plane P. Construction of the projections of this angle φ_1 does not require us to determine either point K or point K_1 (Fig. 168). From two projections of angle φ_1 we find its true magnitude and supplement it to 90°. The angle supplementing the angle found to 90° will be the required angle.

(4) *An angle between two planes.* Two planes P and Q intersect and make four dihedral angles forming equal pairs (Fig. 169). Each is measured by the linear angle obtained if we pass through planes P and Q a third plane R perpendicular to the line of their intersection

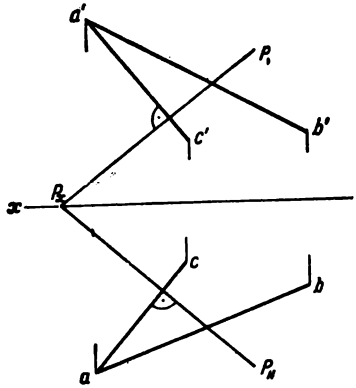


Fig. 168

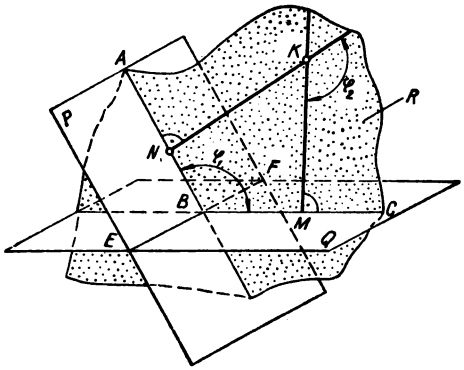


Fig. 169

EF. It may readily be shown that the angle between the two planes is equal to the angle between the perpendiculars to these planes. Indeed, the R plane, defined by two perpendiculars dropped from an arbitrary point of space K on faces P and Q will be perpendicular to them and to edge EF of the dihedral angle. Lines AB and BC (along which plane R intersects planes P and Q) also make right angles with edge EF. Consequently, lines AB and BC are the sides of a linear angle used to measure the dihedral angle. But KN is perpendicular to AB and KM is perpendicular to BC, therefore $\varphi_1=\varphi_2$.

It is advisable to make use of this equation when determining an angle between two planes when each of them is given orthographically by traces (Fig. 170). Dropping perpendiculars from an arbitrary

point K to the given planes, we obtain two lines intersecting in K. The angle between them will then be equal to the required angle made by planes P and Q.

In Fig. 171 the same problem (with faces specified in a different way) is solved by the method of replacing projection planes.

In this case, the common side of two triangles (line AB) serves as the edge of the dihedral angle. The projection AB is transformed into a point by a double replacement of planes. Plane H_1, perpendicular to AB, will be parallel to the sides of the linear angle used to measure the desired dihedral angle φ.

Determining dihedral angles formed by an oblique plane P and projection planes was considered above in connection with the transformation of

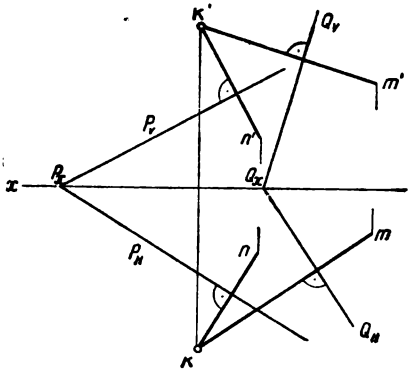


Fig. 170

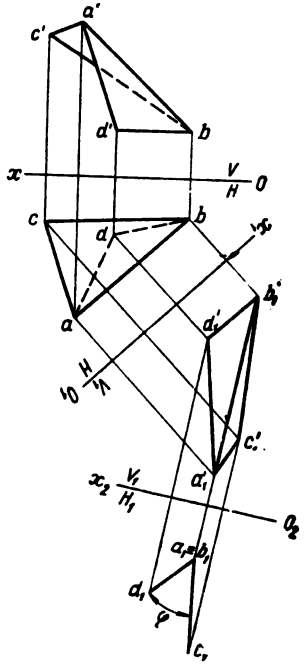


Fig 171

plane P into a projecting plane (see the solution of Problem 3). In Figs. 134-135 angles α (between P and H) and β (between P and V) were found by the method of replacing the planes of projection. Use of the coincidence method for determining angles α and β is shown in Figs. 172 and 173. The edge of the first angle is P_H, that of the second angle is P_V. The plane of the linear angle used to measure the angle between P and H is passed perpendicular to P_H. The linear angle α lies in the horizontal projecting plane Q and is projected on H into a straight line that coincides with Q_H. By bringing the plane Q into coincidence with V in revolution about Q_V, we find the new position of the vertex of the desired angle (point A_1). The angle between the x-axis and the line A_1b' will then be the desired

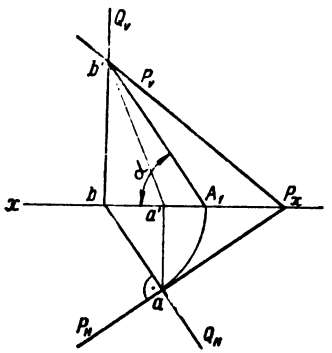


Fig. 172

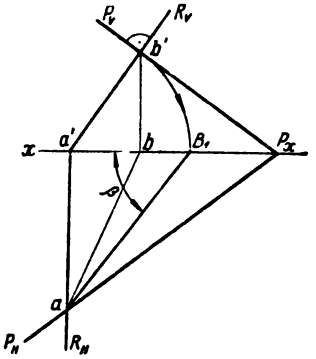


Fig. 173

angle. Angle β is found in a similar fashion. In this case plane R, which contains the linear angle, is constructed perpendicular to P_V and is made to coincide with H. The position of the vertex of angle β coincident with H is labelled B_1.

3. Constructing the projections of a plane figure from given conditions

Let us pose the following problem: to construct a regular triangle ABC located in the given plane P if the radius of the circumscribed circle is R and the centre lies in a given point Ω. The solution of the problem is given in Fig. 174 in the following sequence:

1. Plane P is coincident with H. The coincident vertical trace of the plane, $(P_1)_H$, is constructed with the aid of two points: P_x and N_Ω. The latter point is the vertical trace (coincident with H) of the horizontal line on which point Ω lies.

2. A coincident horizontal line (containing point Ω) is drawn parallel to P_H through point N_Ω.

3. The coincident position of the centre of the circumscribed circle (point Ω_1) is also determined

4. A circle is constructed and a regular triangle $A_1B_1C_1$ is inscribed in it.

5. Through each vertex in the coincident plane are drawn horizontal lines A_1N_A, B_1N_B, C_1N_B (the latter two coincide) to their intersection with $(P_V)_H$. These horizontal lines together with the points they contain will now have to be counterrevolved back to the original position of plane P.

6. From point P_x as a centre we strike an arc of radius equal to P_xN_A to intersection with P_V in point n_A. The other projection n_A of this point will be located on the x-axis.

7. Projections are drawn of the horizontal line on which vertex A is located: the vertical projection through n'_A, the horizontal projection through n_A.

8. A straight line perpendicular to P_H is drawn through vertex A_1 coincident with H. It is along this line that the horizontal projection of point A_1 will move in the counterrevolution of the plane. The horizontal projection a of vertex A will be obtained at the intersection of the straight line perpendicular to P_H with the earlier drawn projection of the horizontal line.

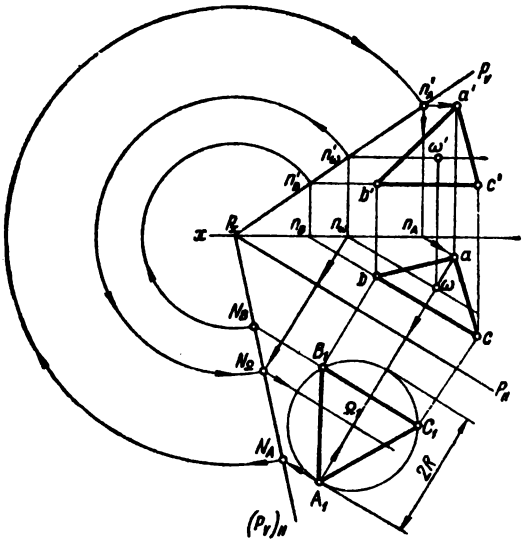


Fig. 174

9. Finally, the perpendicular to the x-axis drawn through a intersects the vertical projection of the horizontal line and will determine a'.

In this way we find the projections of the other vertices of triangle ABC as well.

In this relatively involved construction the most important thing is that through each vertex of the triangle (in the coincident plane) we drew straight lines which were the horizontal lines of plane P; counterrevolution of the plane began with these horizontal lines. The first stage of this counterrevolution culminated in the construction of projections of the horizontal lines; as a result of the second stage, we determined the projections of the vertices of the given plane figure on the earlier prepared projections of the straight lines.

CHAPTER V

POLYHEDRONS

Sec. 33. Fundamental Concepts. Forming the Surfaces of Certain Polyhedrons

A polyhedron is a collection of plane polygons such that each side of one is also a side of another one (but only one other one).

These polygons are called faces, their sides are edges, and the vertices are the vertices of the polyhedron. The collection of all faces of a polyhedron is called its surface. We shall consider only convex polyhedrons, that is, those which are located all to one side of the plane of any one of its faces.

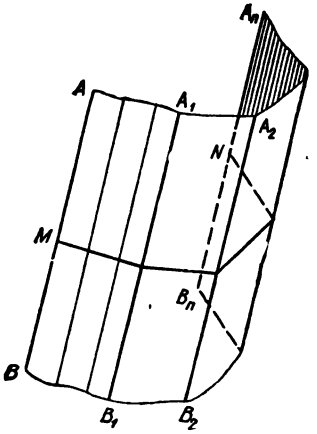


Fig 175

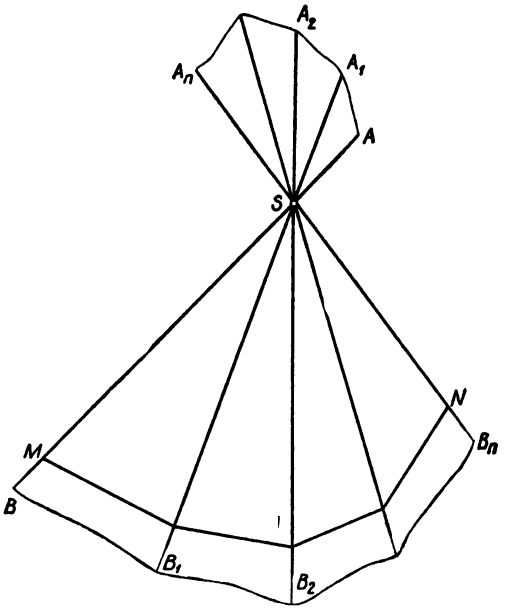


Fig. 176

The formation of the surfaces of certain polyhedrons obeys definite laws. Thus, the lateral surface of prisms (the prismatic surface) is generated by the motion of a straight line AB (the generatrix along the polygonal directrix MN) when AB is always parallel to itself during the motion (Fig. 175). The lateral surface of a pyramid (the pyramidal surface) is obtained by the motion of a straight-line gen-

eratrix AB (that passes through a fixed point S) along the directrix MN (Fig. 176).

In the limiting case when the polygonal directrix becomes a curved-line directrix, the prismatic surface becomes cylindrical, and the pyramidal surface becomes conical.

Sec. 34. Projections of Polyhedrons. Visibility of Edges

The space position of a polyhedron may be specified in a variety of ways: by the coordinates of its vertices, or by the base and the altitude (if the polyhedrons are right and regular), or by one of its faces (if the polyhedron is regular) with the number of faces indicated.

Particularly important is the question of determining the visibility of the projections of the edges of a polyhedron in an orthographic representation.

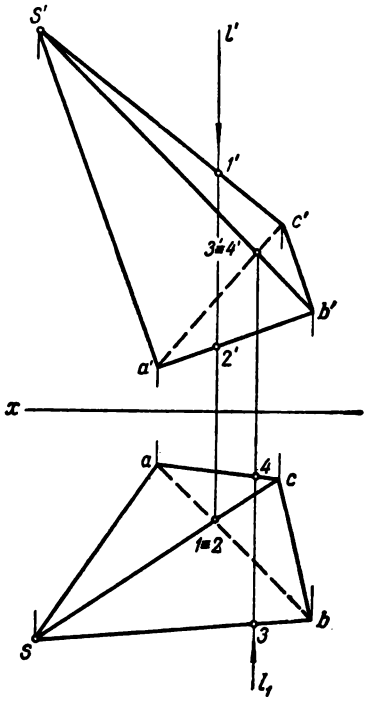


Fig. 177

The outer contour of the projection of a polyhedron will always be visible. But the visibility of the projections of the edges within the contour is determined by means of lines of sight passing perpendicular to H and V through the points of intersection of the projections of the edges.

Fig. 177 indicates the projections of a certain tetrahedron $ABCS$. To determine the visibility of the edges inside the contour of the horizontal projection, take the line of sight L perpendicular to the H plane and passing through the point of intersection of the horizontal projections of the edges SC and AB. (Remember that the line of sight is in a downward direction.) From a consideration of the vertical projection of this line it follows that it will first intersect the edge SC in point I and then AB in point II. Hence, the edge SC will be visible in a top view.

The line of sight L_1, which is directed perpendicular to the V plane and passes through the point of intersection of the vertical projections of edges SB and AC, indicates that edge SB is closer to the viewer than edge AC. This conclusion stems from a consideration of the horizontal projection of the line, which projection intersects the projection of edge SB in point *3* before the projection of edge AC in point *4*.

Sec. 35. A Polyhedron Cut by a Plane

The plane sections of polyhedrons are closed figures, the vertices and sides of which are determined by intersection of a given plane with the edges and faces of the given geometric solid. Hence, to construct sections, one has to find the points of intersection of the edges with a given plane or to construct straight lines along which the plane

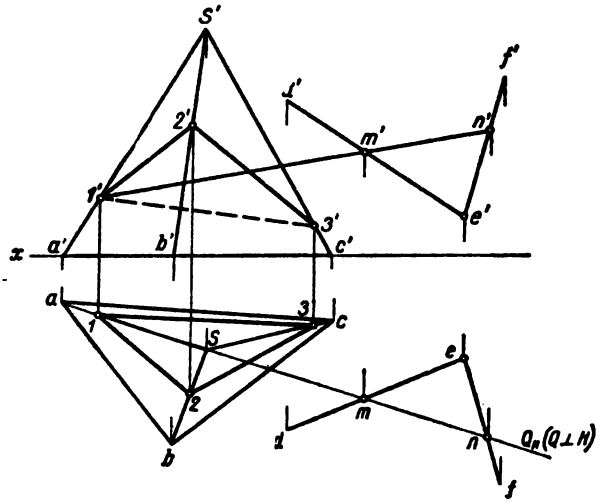


Fig. 178

intersects the faces of the solid. The first method is called the edge method, the second, the face method. These methods will now be applied in a number of concrete cases.

The section of a tetrahedron $SABC$ by a plane given by the intersecting straight lines DE and EF is constructed by the former method (Fig. 178). The orthographic view indicates the auxiliary constructions associated with determining solely the points of intersection of the edge SA with the plane of the lines DE and EF. The horizontal projecting plane Q, which is passed through the edge SA, is indicated in Fig. 178 by only one trace Q_H. The straight line MN is the line of intersection of the given plane and the auxiliary plane. Intersection of the projections $m'n'$ and $a's'$ determines the vertical projection $1'$ of the required point I. The second projection is found in the usual way. In similar fashion we determine the points of intersection of the remaining edges with the given plane.

In Fig. 179 the section of the tetrahedron by the P plane, which is given by traces, is constructed by means of both first and second

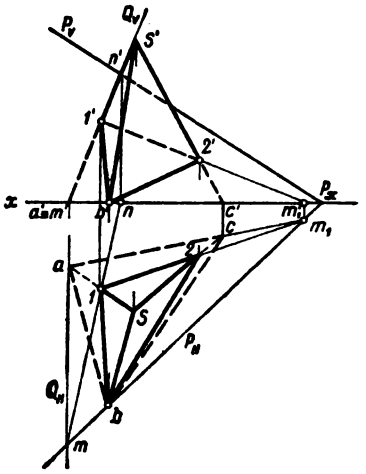


Fig. 179

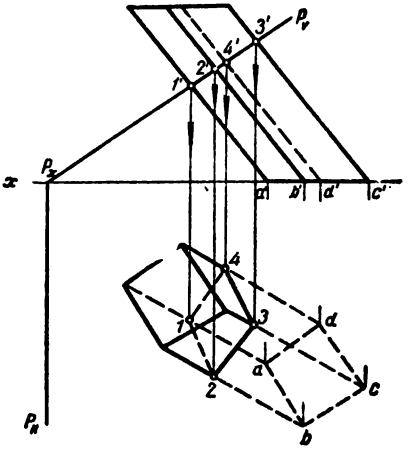


Fig. 180

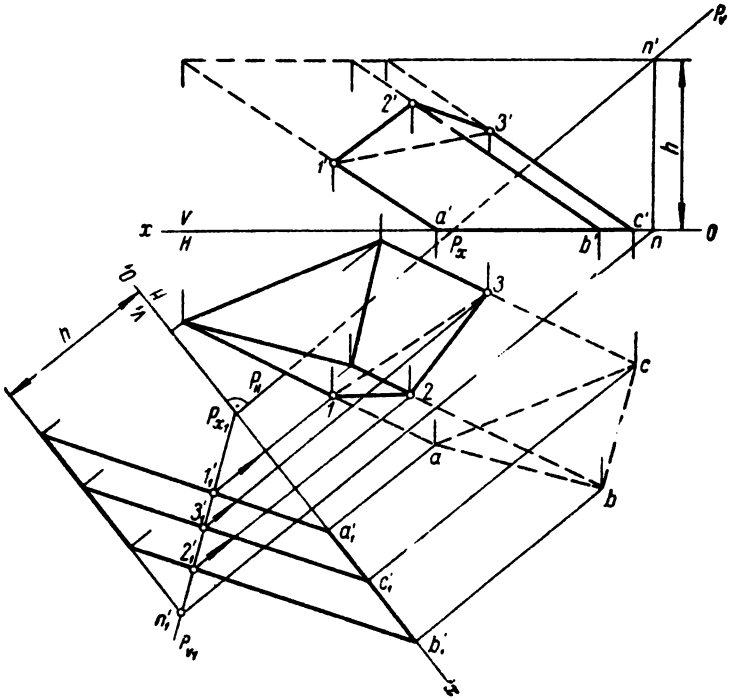


Fig. 181

methods. First, we find point I (the point of intersection of edge SA with the P plane). For this purpose, the vertical projecting plane Q was passed through the edge SA, and line MN, along which P and Q intersect, was constructed. Then we construct the line of intersection of the P plane and face SAC. The first of the two points that determine this straight line will be point I on edge SA; for the second we shall take advantage of the point of intersection of the horizontal traces of the planes under consideration. This will be point M_1 in which the horizontal trace of face SAC (edge AC) intersects P_H. The straight line IM_1 belongs both to the P plane and the face SAC. The line-segment I-II of the constructed line is one of the sides of the section. Joining the extremities of this segment with vertex B we have a closed contour, which is the solution of the problem.

Construction of sections is greatly simplified if the given plane is a projecting plane. This is illustrated in Fig. 180* where a section is constructed of an oblique prism by a vertical projecting plane. In this case, the vertical projection is a straight line coincident with P_V. Drawing perpendiculars through points $1'$, $2'$, $3'$ and $4'$ to the x-axis to intersection with the horizontal projections of the respective edges, we obtain the horizontal projection of the desired section 1-2-3-4.

We can always arrive at this special case so convenient for problem solving by transforming the oblique cutting plane into a projecting plane, as is done in Fig. 181.

Applying the method of replacing planes of projection, we pass from the V/H system to the V_1/H system.

In the new system, plane P became a projecting plane, and the required section is projected on plane V_1 in the form of a straight line coinciding with P_{V_1}.

The reverse transformation of projections from the V_1/H system to the original system permits us to obtain the projections of the desired section on the originally specified projections of the prism.

Sec. 36. Intersection of a Line and a Polyhedron

The problem of determining the points of intersection of a line with the surface of a polyhedron is solved like finding the point of intersection of a line and a plane. In this case too the solution breaks down into three stages:

(1) pass an auxiliary plane Q through a given line EF;

(2) make a section of the polyhedron with plane Q;

(3) determine the points of intersection of the line EF with the contour of the section. They are frequently called "entry" and "exit" points.

* In Fig. 180, as in most of the subsequent figures, the vertices of only one prism base are indicated.

In Fig. 182 the desired points K and L are found as the points of intersection of the line EF with the contour of the plane section. The shape of the plane section is essentially dependent on the position of the auxiliary plane Q relative to the vertex S of the pyramid.

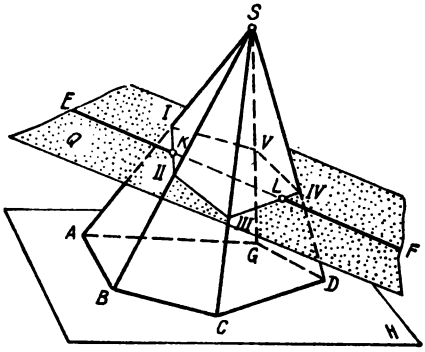


Fig. 182

Indeed, if plane Q passes through the vertex of any polygonal pyramid, we will always have a triangle in the section (the polyhedron is considered convex). Let us agree to call such a plane an elementary cutting plane.

In Fig. 183, just such an elementary cutting plane Q is passed through line EF. The space position of this plane is determined by the given line EF and the vertex S of the pyramid. Trace Q_H of plane Q intersects the base of the pyramid at points I and II. Connecting them with the vertex S, we get the contour of the section. What remains now is to note points K and L, in which line EF intersects the constructed plane section $ISII$.

If in place of a pyramid we have a polygonal prism, then an elementary cutting plane must pass through line EF parallel to the lateral edges of the prism (Fig. 184). This plane will intersect any polygonal prism in a parallelogram. The position of the elementary cutting plane Q in Fig. 184 is determined by the given line EF and the line EM_2 that intersects it and is parallel to the lateral edges of the prism. We construct a section of the prism cut by plane Q and note the desired points K and L.

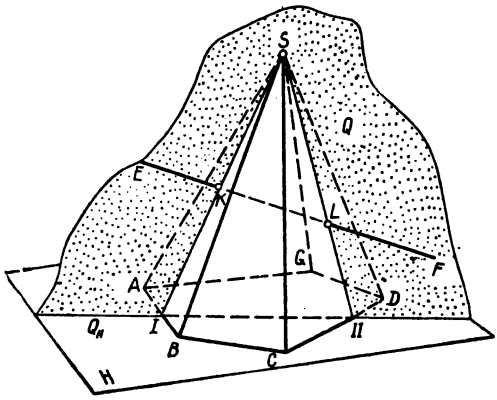


Fig. 183

There are a number of peculiarities in the orthographic solutions given in Figs. 185 and 186. In the former, to construct the horizontal trace of the elementary cutting

plane we had to draw a line through the vertex S of the pyramid. This line intersects the given line in an arbitrary point, say, E. Thus, the elementary cutting plane Q is now determined by the two intersecting straight lines EF and SE. Constructing the horizontal traces of each of them, we draw Q_H. It is then advisable to construct both projections of the section of the pyramid (triangle $ISII$) and determine (independently) the projections of the desired points K and L. Perpendicularity to the x-axis of the lines of recall will serve as a check on the accuracy of the solution obtained.

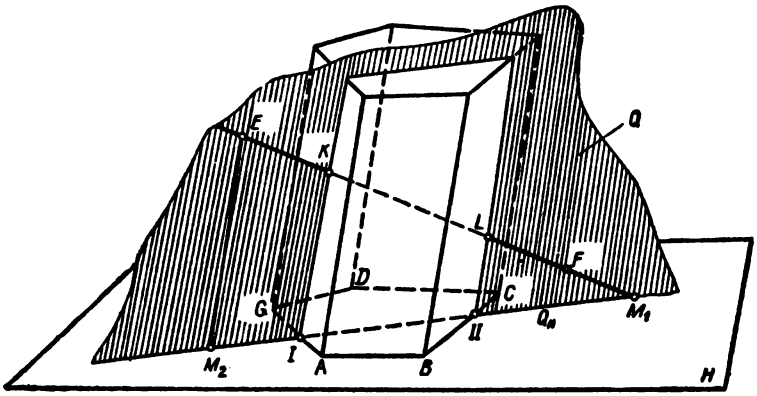


Fig. 184

To construct an elementary cutting plane (see Fig. 186) through an arbitrary point of a given line EF (say point E) draw a straight line EM_1 parallel to the lateral edges of the prism. The horizontal traces M_1 and M_2 of both intersecting lines EF and EM_1 determine the trace Q_H of the elementary cutting plane. Noting points I and II in which Q_H intersects the base of the prism, we construct the horizontal projection of the section of the prism (the parallelogram *1-3-4-2*). The sides of this parallelogram *1-3* and *2-4* intersect *ef* in points *k* and *l*, which are the horizontal projections of the required points of entry and exit. Their second projections are obtained by means of the vertical projection of the parallelogram *1'-3'-4'-2'*.

In the foregoing cases, the base of the polyhedron was located on one of the planes of projection, thus simplifying the solution as compared with the case when the base of the polyhedron does not lie in the H or V plane. In the latter case it is necessary to construct not the trace of the elementary cutting plane Q, but the line of intersection of the same auxiliary plane Q with the base of the polyhedron. We then note the intersection of this line with the con-

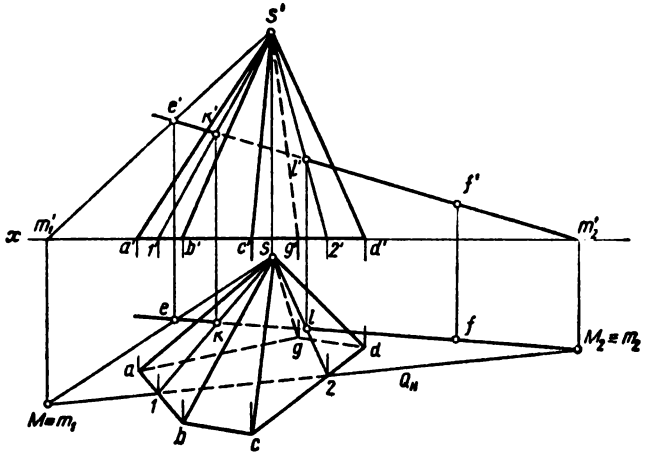


Fig. 185

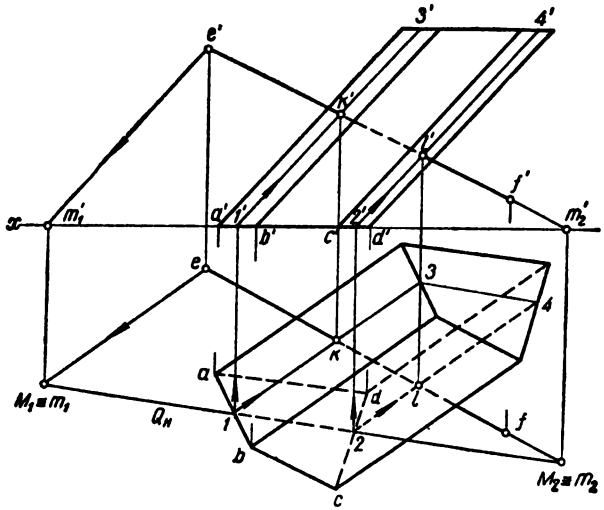


Fig. 186

tour of the base of the pyramid or prism and construct the projections of the plane section of the polyhedron. The section contour intersects the given straight line and enables us to find the required points.

The solution is very cumbersome, so if the base of the polyhedron does not lie in the projection planes, it is advisable to take a projecting plane as the auxiliary plane. In Fig. 187 the vertical projecting plane Q is passed through line EF and the projections of the section (by plane Q) of a triangular prism are constructed. The contour of the

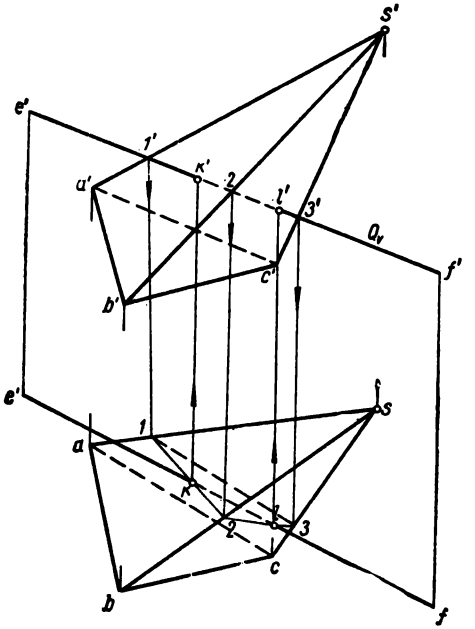


Fig. 187

horizontal projection *1-2-3* of the constructed section intersects *ef* in points *k* and *l*, which are the horizontal projections of the required points. Their vertical projections *k'* and *l'* are found by means of the lines of recall.

Note that projecting planes are best used for any locations of the bases of triangular prisms and pyramids. But if the number of lateral faces of the polyhedron exceeds three and the base is located in the *H* or *V* plane, then the solution is better obtained by using an elementary cutting plane.

Elementary cutting planes are particularly convenient when determining points of intersection of a straight line and a cone and cylinder (which we shall consider in Sec. 46) and also in the construction of the line of intersection of two polyhedrons.

Sec. 37. The Mutual Intersection of Polyhedrons

Two polyhedrons can intersect along one or two closed polygonal lines, for the construction of which we first find the intersection points of the edges of one polyhedron with the faces of the other, then the edges of the second with the faces of the first. Connecting, in appropriate fashion, the points thus obtained, we construct the required polygonal line, each segment of which is the straight line of intersection of two faces: a face of the first polyhedron and a face of the second.

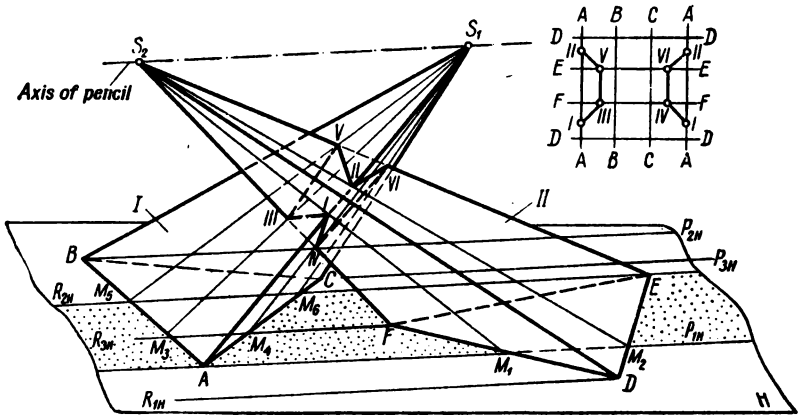


Fig. 188

Thus, construction of the intersection line of two polyhedrons reduces to the solution of a problem on the intersection of a straight line and a polyhedron. In Sec. 36 it was shown that for a rational solution of this problem projecting planes should be used in some cases and elementary cutting planes in others. The latter should be resorted to if the bases of both polyhedrons are located in a single plane. The constructions become simpler if this plane is one of the projection planes. Let us examine the use of the method of elementary cutting planes in the construction of the intersection line of pyramids and prisms.

1. *Intersection of two pyramids.* The elementary cutting plane must intersect both pyramids in triangles. This condition is fulfilled if the plane passes through the line S_1S_2 connecting the vertices of the pyramids (Fig. 188). All cutting planes used to construct the line of inter-

section of the polyhedrons form a pencil, the axis of which is the line S_1S_2. Let both pyramids shown in Fig. 188 have the same altitude. Then the axis of the pencil of elementary cutting planes will be parallel to H. Being the common horizontal line of the auxiliary planes, the line S_1S_2 determines the direction of their horizontal traces, which in the given case form a pencil of parallel straight lines. We determine the points of intersection of the edges of the first pyramid S_1ABC with the faces of the second one, S_2DEF. Through edge S_1A (Fig. 189) draw the elementary cutting plane P_1. Its position is determined by two intersecting lines S_1S_2 and S_1A. P_{1H} will pass

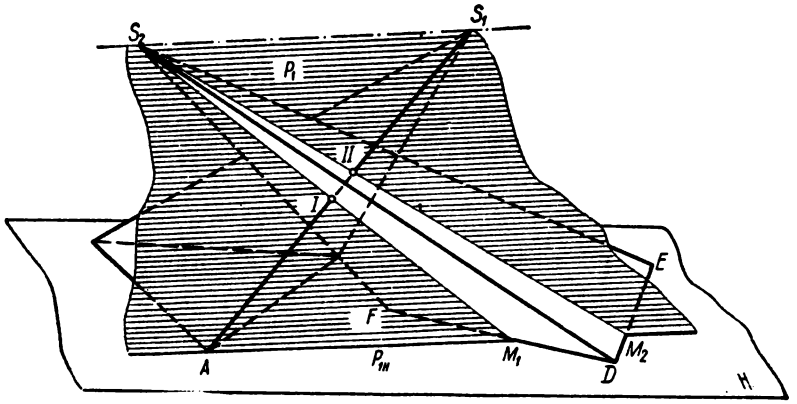


Fig. 189

parallel to S_1S_2 through point A, which is the horizontal trace of edge S_1A. From the drawing it is evident that P_{1H} intersects the base of the second pyramid at points M_1 and M_2, which when joined with vertex S_2 yield a plane section of the pyramid: the triangle $M_1S_2M_2$. The sides of the triangle M_1S_2 and M_2S_2 intersect the edge S_1A and determine the desired points I and II. This construction is shown in Fig. 188. Passing auxiliary planes through edges S_1B and S_1C, we note that the traces of the planes P_{2H} and P_{3H} do not intersect the base of the second pyramid.

Consequently, the planes P_2 and P_3 and, hence, edges S_1B and S_1C do not intersect the faces of pyramid S_2DEF. Having completed this investigation of the edges of the first polyhedron for intersection with the faces of the second, we begin the next stage: to determine the intersection points of the edges of the second pyramid with the faces of the first. Of the three planes R_1, R_2 and R_3, the first does not cut the pyramid S_1ABC. Which means that edge S_2D, through which R_1 passes, does not intersect the faces of the first pyramid. The points

of entry and exit of the other two edges are found with the aid of plane sections: triangles $S_1M_3M_4$ and $S_1M_5M_6$.

These are points *III*, *IV*, *V* and *VI*, of which the first two lie on edge S_2F and the second on S_2E. The question now is how to join these points.

Since each segment of the desired polygonal space line is an intersection line of the faces of two different pyramids, we have to join only those points which simultaneously belong to the same faces of the intersecting polyhedrons. Guided by this fact, in Fig. 188 we construct the closed line *I-III-V-II-VI-IV-I*. Thus, the points *I* and *III* of this polygonal line are joined because they both lie in the common faces S_1AB of the first pyramid and S_2DF of the second pyramid. Points *III* and *V*, which lie in the common faces S_1AB and S_2EF, determine the next segment, etc.

To avoid errors in complicated cases of the intersection of polyhedrons, it is advisable to make use of a specially constructed grid (shown in Fig. 188). The number of vertical lines of the grid must be greater (by unity) than the number of lateral edges of the first polyhedron, and the number of horizontal lines must exceed by one the number of lateral edges of the second polyhedron.

In our case of the intersection of two triangular pyramids, we draw four vertical and four horizontal lines.

The vertical lines will correspond to the edges of the first polyhedron, pyramid S_1ABC for instance, and the horizontal lines, to the edges of the second one S_2DEF, the first and last lines of each direction corresponding to one and the same edge. In our case, edge S_1A is repeated on the vertical lines and edge S_2D is repeated on the horizontal lines. In the grid, each edge is denoted by a single letter (vertex S_1 and vertex S_2 are not indicated). The intervals between the vertical lines will correspond to the respective faces of the first polyhedron, and the intervals between the horizontal lines, to the faces of the second one.

On the grid we indicate the points obtained in Fig. 188, noting on which edge of one polyhedron and on which face of the other the given point lies. For instance, point *I*, which is the intersection point of edge S_1A with face S_2DF, is depicted on the vertical line *A* inside the band bounded by the horizontal lines *D* and *F*. Point *II*, in which edge S_1A intersects the face S_2DE, is recorded on line *A* in the interval between lines *D* and *E*. Point *III* is shown on line *F* between lines *A* and *B*, and so forth. The points belonging to the edges that have been repeated on the grid are also depicted twice. Then the points indicated on the grid and located on the sides of a given cell are joined in pairs. In this way we get the polygonal line *I-III-V-II-VI-IV-I*. Use this sequence to join the points found in Fig. 188. The only thing that remains is to indicate the visibility of the segments of the constructed

polygonal line. The segment along which two visible faces intersect is visible. If even one of the faces is hidden, then the segment of the desired line located on this face will be hidden.

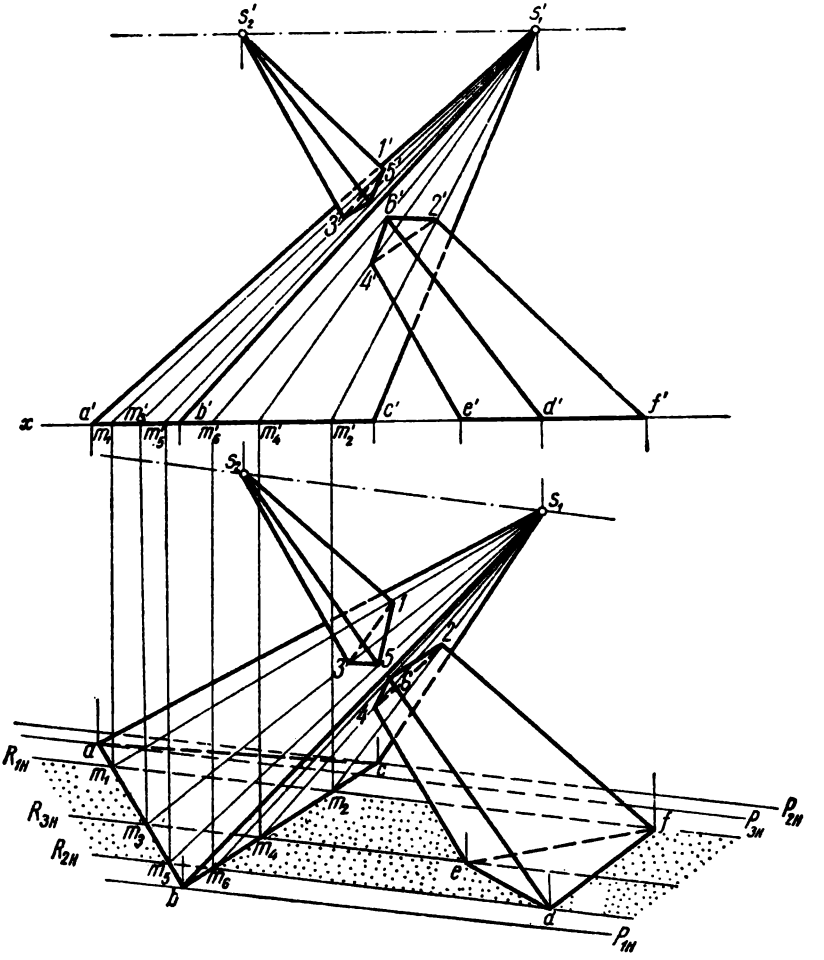


Fig. 190

An orthographic solution of the intersection of two pyramids of the same altitude is shown in Fig. 190. Here too the axis of the pencil of elementary cutting planes is their horizontal line. For this reason, the horizontal traces of the auxiliary planes are parallel to S_1S_2. A characteristic feature of the example considered in Fig. 190 is that

the intersection line of the pyramids has broken down into two closed polygonal lines: two triangles. To determine the vertices of the required polygonal line, an elementary cutting plane was passed through each edge; the section of the polyhedron cut by this plane was constructed, and, finally, the points of intersection were noted of the edge under investigation with the constructed plane section. Thus, through edge S_2F we pass plane R_1, the horizontal trace of which, R_{1H}, passes through the like trace of the edge (point f) parallel to S_1S_2. Triangle $S_1M_1M_2$ is a section of pyramid S_1ABC cut by plane R_1.

Constructing both projections of the triangle $S_1M_1M_2$, we note the points of intersection of the projections of its sides with the like projection of edge S_2F. We thus obtain the projections of points I and II, which are the points of entry and exit of edge S_2F. In a similar way we find the remaining points. Planes P_1, P_2 and P_3, which pass through the edges of pyramid S_1ABC, do not intersect the second polyhedron. Hence, edges S_1A, S_1B and S_1C do not participate in the intersection.

Some of the questions that arise in connection with the intersection of polyhedrons may be resolved without constructing the desired line. For example, we can establish which edges of each of the polyhedrons will participate in the intersection and along which line the intersection will occur (along one or two closed lines). Answers to these questions enable us to obtain what are called regional planes.

Let us agree to give the name regional plane of one of two intersecting polyhedrons to a plane originating from a pencil of elementary cutting planes that passes through an edge of the given polyhedron and does not intersect the latter.

It is always possible to isolate two regional planes from a pencil of elementary cutting planes for each polyhedron participating in the intersection.

In Fig. 188 planes P_1 and P_2 are regional planes of pyramid S_1ABC. Passing, respectively, through edges S_1A and S_1B, not one intersects pyramid S_1ABC. Bounding the polyhedron on two sides, both planes form a dihedral angle, inside of which lies the pyramid in question. The term "regional planes" reflects to some extent the fact that these planes isolate a "region" of space that contains the given polyhedron.

In the same figure, planes R_1 and R_2 are regional planes of the pyramid S_2DEF.

The same letters and subscripts are used in Fig. 190 to denote the regional planes of the polyhedrons. Let us now establish the relative positions of the regional planes and their traces for cases when the polyhedrons are intersected along one and two closed lines; we shall exclude such particular cases as when the bases intersect or when the vertex of one pyramid is located inside the other polyhedron.

In Figs. 191, 192, and 193 we consider possible cases of the relative positions of two pairs of planes belonging to a single pencil.*

In the foregoing drawings, let planes P_1 and P_2 be regional planes of the first polyhedron, and R_1 and R_2 regional planes of the second one.

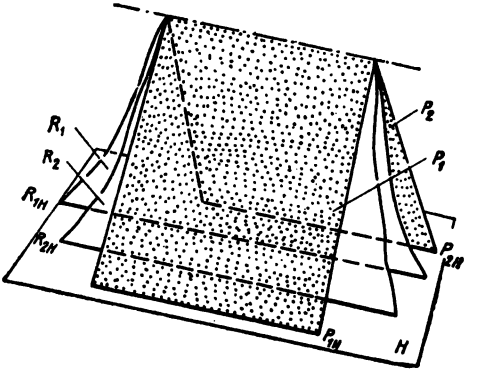


Fig. 191

In Fig. 191 the dihedral angle formed by the two planes R_1 and R_2 lies inside an angle whose faces are P_1 and P_2. This means that all edges of the second polyhedron will be intersected by the faces of the first one. The second pyramid pierces the first straight through. The

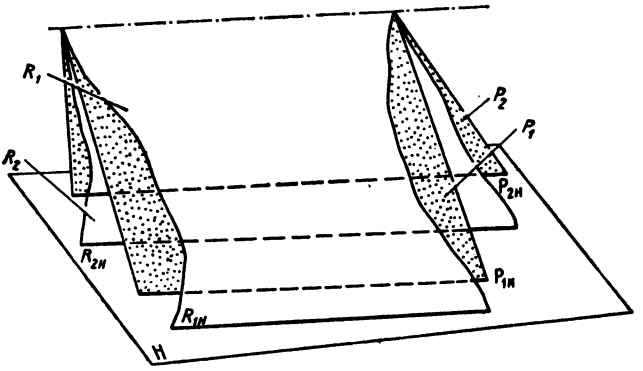


Fig. 192

line of intersection breaks down into two closed polygonal lines. In this case the traces of the regional planes of one polyhedron appear

* It is left to the student to work out cases when one or two pairs of regional planes coincide.

between the traces of the regional planes of the other one (Fig. 194). Let us now examine Fig. 192 where the regional planes of two polyhedrons alternate. The intersection will involve those parts of the pyramids that are located in the common dihedral angle formed by the planes P_1 and R_2. This time, one of the pyramids only partially "cuts" into the other.

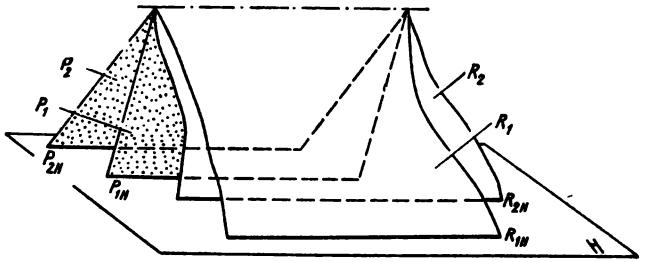


Fig. 193

The polyhedrons will intersect along one closed line. The test of this case is the alternation of the traces of the regional planes (Fig. 195).

Finally, a third case is possible in the relative positions of the regional planes (see Fig. 193) when the dihedral angles formed by each pair of regional planes do not have any common elements, and so the polyhedrons located inside these angles cannot have any common points. They do not intersect. These conclusions hold for any position

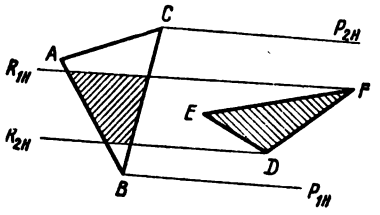


Fig. 194

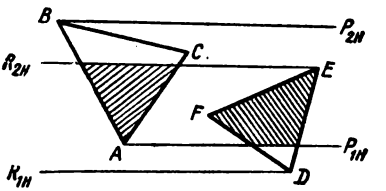


Fig. 195

of the axis of the pencil of cutting planes relative to the projection planes and for any other combination of intersecting polyhedrons (two prisms, a pyramid and a prism).

Only when the axis of the pencil S_1S_2 is not parallel to the projection plane H will the horizontal traces of both regional and all elementary cutting planes pass through one point: the horizontal trace of line S_1S_2. The bases of pyramids (with unequal altitudes) intersecting

along two closed lines and the positions of the traces of their regional planes are shown in Fig. 196.

The same for the case of intersection along a single closed line is depicted in Fig. 197.

An instance of the construction of such a line is given in Fig. 198. Joining the vertices of the pyramids with a straight line we get the axis of a pencil of elementary cutting planes and find its horizontal trace M. Through point M draw the traces of regional planes P_{1H}, P_{2H} of the first polyhedron and R_{1H}, R_{2H} of the second one. Alternation of traces signifies that the desired line will be a single closed line.

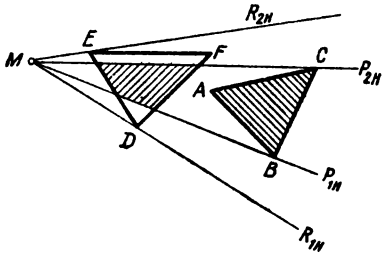


Fig. 196

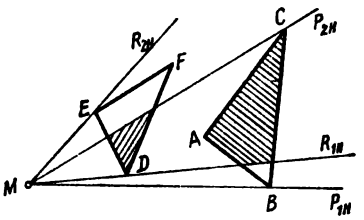


Fig. 197

The intersection will involve those edges, the horizontal traces of which are inside and on the sides of the angle formed by the traces P_{1H} and R_{2H}. In our case there are three such edges: S_1B, S_1C and S_2F. The points of entry and exit of each of them are found by means of elementary cutting planes.

For example, through edge S_1B we pass plane P_1, which is determined by the straight lines S_1S_2 and S_1B, that is, by the axis of the pencil of elementary planes and by the given edge. The section of pyramid S_2DEF cut by this plane will be the triangle $S_2M_1M_2$. The points of intersection I and II of edge S_1B with the contour of triangle $S_2M_1M_2$ will be the vertices of the desired line. The remaining points are determined in exactly the same way. The grid shown in Fig. 198 helps us to establish the sequence for joining these points.

2. *Intersection of a pyramid and a prism.* An elementary cutting plane should cut a pyramid in a triangle, a prism in a parallelogram. This condition will be fulfilled if the axis of the pencil of elementary cutting planes passes through the vertex of the pyramid parallel to the lateral edges of the prism. Fig. 199 shows one of the planes of the pencil passing through edge SA of the pyramid. The position of this plane is determined by two lines SM and SA. From the drawing it is evident that P_H cuts the prism base in points M_1 and M_2, through which we draw lines parallel to the lateral edges of the prism. These lines of a plane section of the prism intersect edge SA and yield points

I and *II*. The foregoing construction is repeated in the orthographic representation of Fig. 200 both for edges *SA* and *SC* of the pyramid and for edge *D* of the prism. The remaining edges do not participate in the intersection. The grid enables us to establish the sequence for

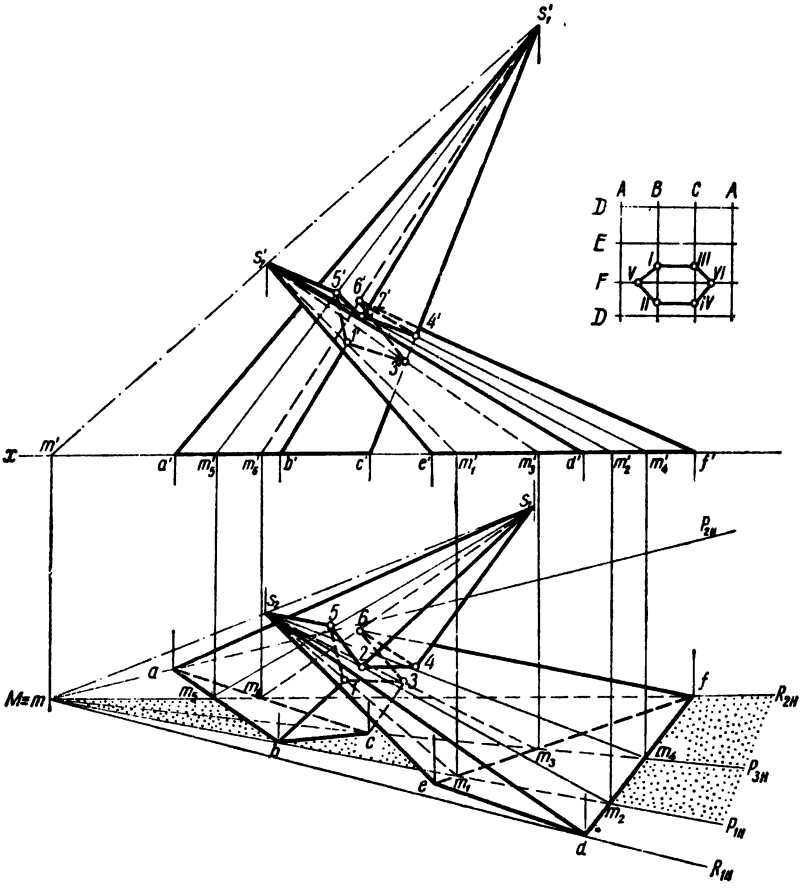


Fig. 198

joining the points found. The line of intersection turned out to be a single closed line, as might have been foreseen prior to construction, since we have an alternation of horizontal traces of regional planes P_{1H}, P_{2H} of the pyramid and R_{1H}, R_{2H} of the prism.

3. *Intersection of two prisms.* An elementary cutting plane should intersect each of the prisms in a parallelogram. This condition will

be fulfilled if the auxiliary cutting planes are parallel to the edges both of the first and second prism. In the general case, the lateral edges of one prism are skew lines with respect to the lateral edges of the other. Let us agree to call the plane parallel to two skew lines a plane director. For two skew lines we can construct any number of such planes. Since each of them is determined by two skew lines respectively parallel to the given skew lines, they are all mutually parallel.

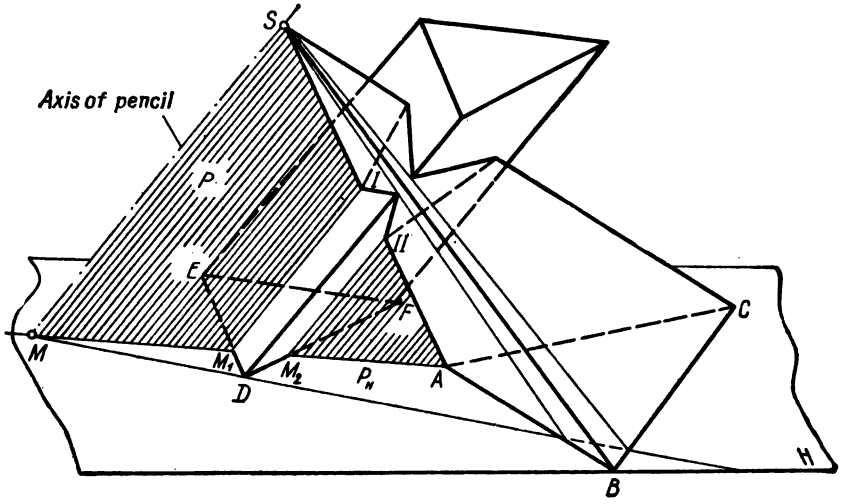


Fig. 199

The elementary cutting planes for the two prisms will also be parallel to each other. The directions of their traces are ordinarily found as follows: through some point K (Fig. 201) draw lines KM and KM_0 parallel to the edges of the prisms. These lines determine one of the plane directors P for the edges of the intersecting prisms. The elementary cutting planes must be parallel to P and their horizontal traces will be parallel to P_H. The latter is defined by the horizontal traces M and M_0 of the lines drawn through point K.

Draw parallel to P_H the traces of the regional planes P_{1H}, P_{2H} of prism I and R_{1H}, R_{3H} of prism II. Since the second set of traces lie between the first set, the polyhedrons will intersect along two closed lines. The edges of prism I, whose horizontal traces are situated beyond the limits of a band bounded by R_{1H} and R_{3H}, will not participate in the intersection. The points of entry and exit of the edges of prism II are found in the customary way: an elementary cutting plane is passed through each of them, and then these planes are used to cut sec-

tions of prism *I*. The points of intersection of the plane sections with the respective edges will be the required vertices of the two closed lines.

It will be noted that while planes R_1 and R_3 are regional planes for one prism, they will be elementary cutting planes with respect to the other.

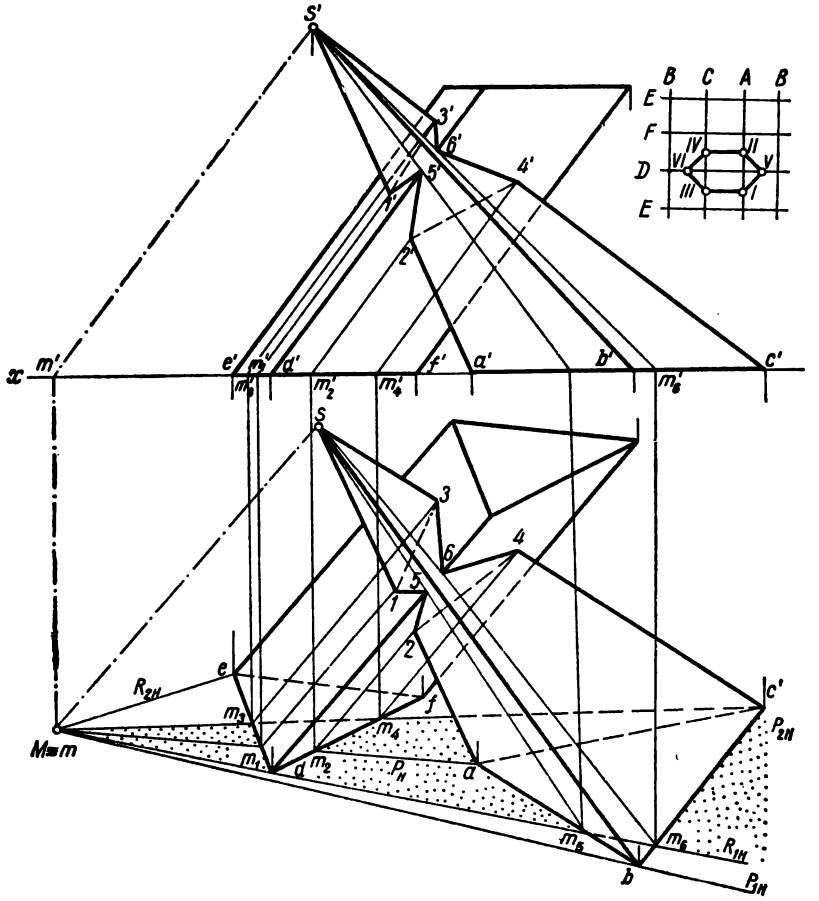


Fig. 200

It is simple to join the points thus found since all even points are on one face, *BC*, and all odd points are on face *AB*. The same result will be obtained if we take advantage of the grid shown in the same figure.

In concluding this study of examples of the expedient use of elementary cutting planes for the construction of intersection lines of pyramids and prisms, we note that it is best to resort to elementary cutting planes when the bases of the two polyhedrons are located in a single plane. If this plane is some oblique plane P, the foregoing

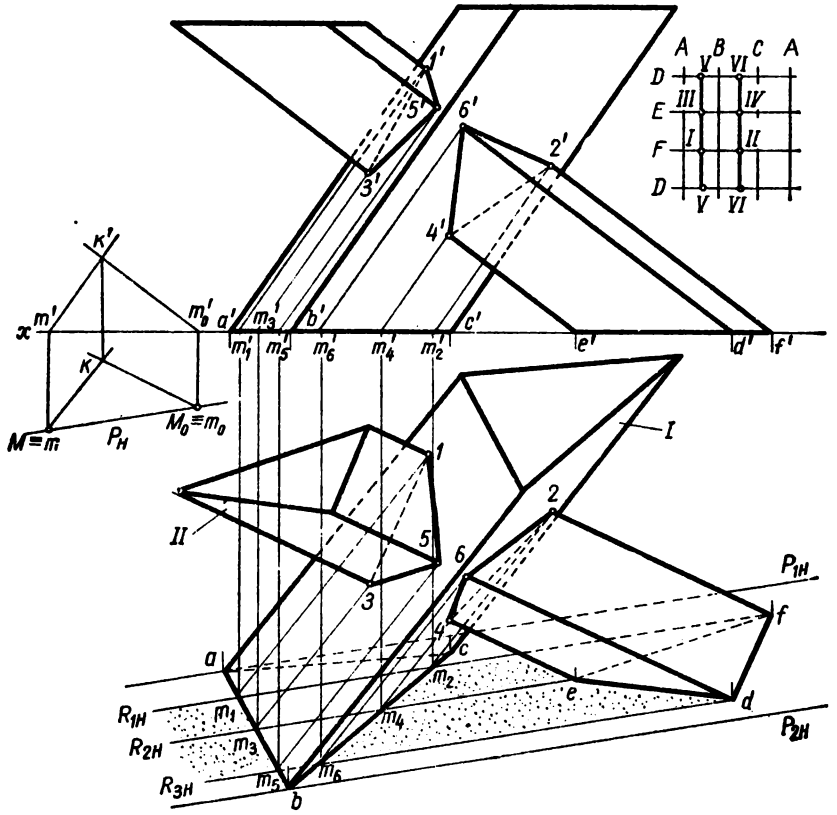


Fig. 201

constructions will become somewhat more complicated due to the fact that in place of the traces of elementary cutting planes we shall have to construct (in the projection planes) lines of intersection of auxiliary planes with the base plane P.

Let us now examine Fig. 202 that contains two polyhedrons whose bases do not lie in one plane. Here, we shall have to seek the points of intersection of the edges of each polyhedron with the faces of the other by means of projecting planes and not elementary cutting planes.

Fig. 203 repeats pyramid $SABC$ and the two prism edges DD_1 and FF_1. An auxiliary vertical plane is passed through each of them. Plane U, which passes through edge DD_1, intersects the pyramid in the triangle LMN, the vertical projection $l'm'n'$ of which is indicated by cross-hatching.

Triangle $L_1M_1N_1$ is the section of the pyramid cut by plane U_1 passed through FF_1. Edge DD_1 does not participate in the intersection since its vertical projection $d'd_1'$ does not have any common points with the projection of the plane section $l'm'n'$.

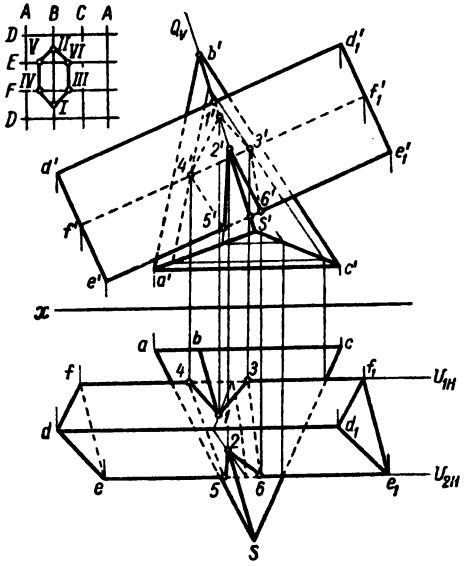


Fig. 202

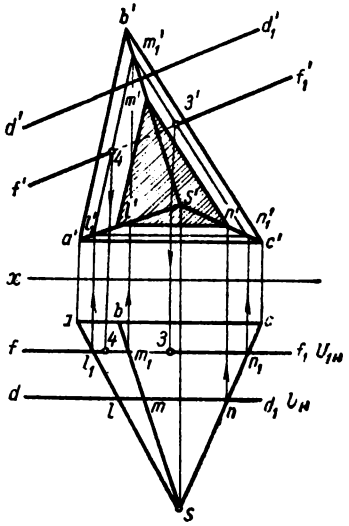


Fig. 203

The vertical projection of triangle $l_1'm_1'n_1'$ intersects $f'f_1'$ and determines the vertical projections $3'$ and $4'$ of the points of entry and exit of edge FF_1. These same constructions are made for edge FF_1 and the other edges in Fig. 202, where an auxiliary plane U_2 (vertical) is passed through the prism edge EE_1 and a vertical projecting plane Q is passed through the pyramid edge SB. Points V and VI are found on edge EE_1, and points I and II are found on edge SB. There is no need to pass planes through the pyramid edges SA and SC. These edges do not intersect the prism faces since their vertical projections do not intersect the like projections of the prism.

We join these points in accord with the grid shown in Fig. 202.

We conclude this section with an instance of the intersection of polyhedrons when the faces of one are perpendicular to some plane of

projection. Since the faces of the triangular prism shown in Fig. 204 are perpendicular to H, the horizontal projections of the intersection points of the pyramid edges are noted orthographically without any

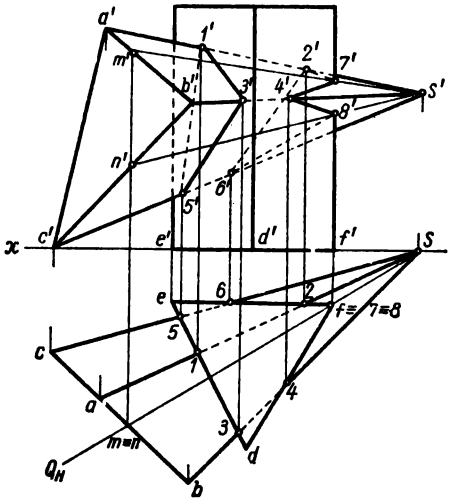


Fig. 204

auxiliary constructions. We find the vertical projections of these points by drawing lines of recall. To determine points *VII* and *VIII* it was necessary to pass the auxiliary horizontal projecting plane Q through only one prism edge F.

CHAPTER VI

CURVED LINES

Points, straight lines and planes are called elementary geometric images in the sense that they suffice to construct all other geometric images.

Taking a point as an elementary geometric image, *we may regard any line as the locus of a point moving through successive positions, or the path of a point. In certain cases a line may be defined as the locus of points that have some common property.* For example, an ellipse is defined as the locus of points, the sum of the distances of each of which to two given points F_1 and F_2 is a constant: $F_1A + AF_2 = \text{const}$ (Fig. 205), where A is an arbitrary point belonging to the ellipse

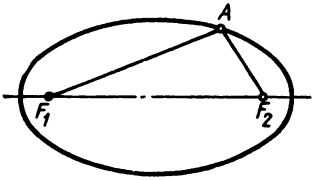


Fig. 205

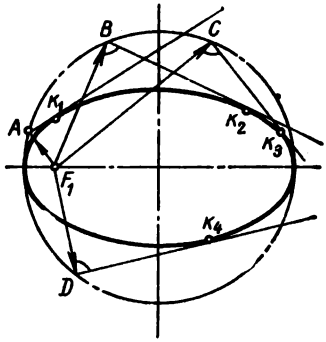


Fig. 206

If for the elementary image we take a straight line, then any line may be regarded as the envelope of its tangents.

Occasionally a family of straight lines whose envelope is a line is the locus of straight lines that obey given geometric conditions. The ellipse may also be defined as the envelope of straight lines perpendicular to the radius vectors of a circle The latter are drawn from a point inside the circle and each straight line of the family passes through the extreme point of the radius vector, which point lies on the given circle.

Fig. 206 shows a circle and a point F_1 inside it, from which are drawn radius vectors F_1A, F_1B, F_1C and so forth. The straight lines

passing through points A, B, C, etc. (which are perpendicular, respectively, to F_1A, F_1B, F_1C, etc.), form a family of straight lines whose envelope will be an ellipse. The points of tangency of the ellipse and the lines of the family are designated as K_1, K_2, K_3, etc.

The points of a curved line may also be determined by the law of its motion.

In mechanics, the equations of motion of a point in coordinate form are

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

Assigning to the parameter t (time) various values, we determine the coordinates of the moving point and thus find its position in space.

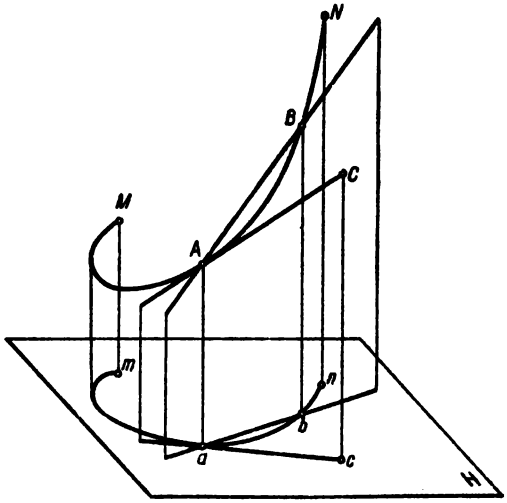


Fig. 207

The locus of all these points forms a certain line whose parametric equations are the equations of motion of a point.

These are not the only methods of specifying curves in analytic geometry.

Not every line is expressible in analytic form. Those that are not are given graphically in a drawing.

It is also possible to depict graphically any line that is generated in accord with specific geometrical conditions.

In descriptive geometry curved lines are studied on the basis of their projections. The construction of the projections of lines depends essentially upon whether all the points of a given curve lie in one

plane or not. If all points of the curve lie in one plane, we have a *plane curve*, examples of which are circles, ellipses, parabolas, hyperbolas, cycloids, and so forth.

A curved line that does not lie entirely in the plane is a space curve (line of double curvature).

These curves are exemplified by helices, which find numerous engineering applications.

We shall point out certain properties of the projections of plane and space curves.

If a point A lies on curve MN (Fig. 207), the projection a of this point lies on the projection mn of the curve.

The secant* and tangent to the curve are projected, respectively, into the secant and tangent to the projection of the curve.

Now pass a secant AB (Fig. 207) through point A of curve MN. The projection of the secant is determined by the points a and b which lie on the projection mn of the curve under consideration. Line ab intersects the projection of the curve and is a secant with respect to mn.

The tangent line AC may be regarded as the limiting position of the secant which the latter occupies as it approaches the intersection points A and B until they merge in the single point A. In this case their projections will also coincide: the projection of the secant will turn into the line ac tangent to the projection mn of the curve.

Sec. 38. Projections of Plane Curves

When constructing the projections of a plane curve it is necessary to indicate the so-called characteristic points, which include the singular points of the curve and also the nearest and farthest points with respect to the projection planes.

A point M (Fig. 208) that may be enclosed in a rectangle (even though a very small one) so that the portion of the curve within it is a simple line-segment** is called an ordinary point of the curve. All other points are called singular points (A and B in Fig. 208).

Fig. 209 shows certain singular points of plane curves: 1) node A, or a point of self-intersection (Fig. 209*a*) at which the curve has two different tangents; 2) a cusp of the first kind B (Fig. 209*b*), in which the curve approaches a point as two branches which have a common tangent at point B and are located on different sides of the tangent; 3) a cusp of the second kind C (Fig. 209*c*), in which the curve ap-

* A secant is a straight line that cuts a curved line in one, two or more points.

** An elementary line-segment is a locus of points the coordinates of which satisfy (in at least one rectangular Cartesian system) the equation $y=f(x)$ when $x_1 \leqslant x \leqslant x_2$, where x_1 and x_2 are two fixed values and $f(x)$ is assumed to be single-valued, continuous and differentiable a sufficient number of times.

proaches a point as two branches that have a common tangent at point C which is located (in the neighbourhood of point C) on one side of the two branches of the curve.

Fig. 210 is a construction of the horizontal projection of a curve lying in the P plane given by the traces P_V and P_H. In this case the vertical projection of the line is also given.

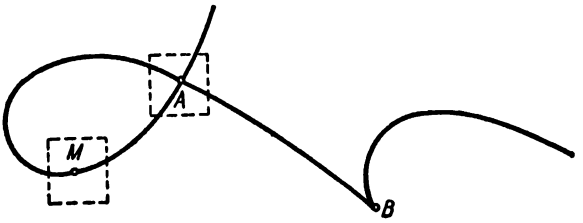


Fig. 208

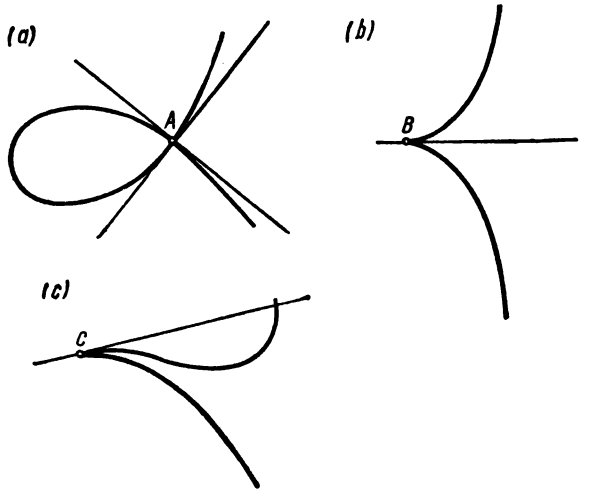


Fig. 209

Essential to the construction is the solution of a problem described in Sec. 18, where the horizontal projection a of point A was determined from the vertical projection a' of A lying in the given plane.

Let us assume that through point A of the curve in question we draw a horizontal line AN in plane P. The vertical projection $a'n'$ must pass through the given like projection of the point a'. The second

projection of this horizontal line is drawn parallel to P_H through the horizontal projection of its vertical trace (point n).

We use a line of recall to determine the desired horizontal projection a of point A. Similarly, we find the horizontal projections of points B, C, D, and E. It will be noted that point E is the lowest point of the curve. The horizontal projection of E was obtained with the aid of the horizontal line EN_3, which is a tangent to the given curve.

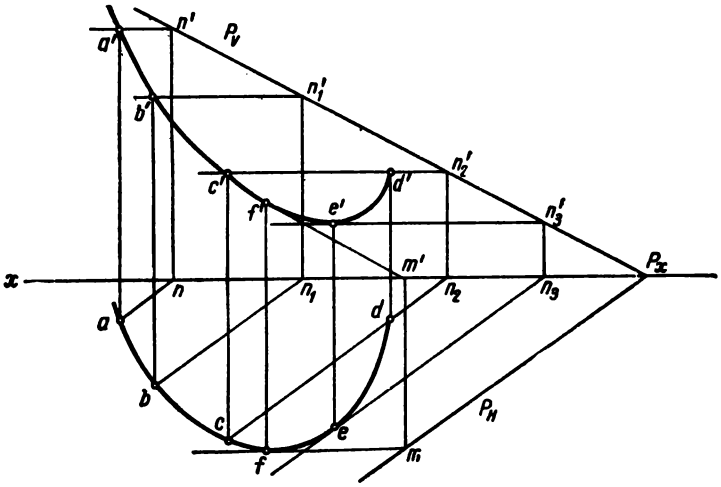


Fig. 210

The tangent FM drawn parallel to the vertical trace of plane P_V determines point F, the most distant point from the V projection plane.

We consider yet another example of the construction of projections of a plane curve (a circle situated in an oblique plane P) with centre in point Ω (ω, ω') and of radius R (Fig. 211).

In our case the orthogonal projections of the circle will be ellipses; here, the projection of the diameter of the circle that is parallel to the given projection plane will be the major axis of each ellipse. All other diameters will be projected with distortion (reduction in size). Consequently, the projections of the circle diameters that are coincident with the horizontal line and vertical line of the P plane will be the major axes of the ellipses, which are, respectively, the horizontal and vertical projections of the circle.

In Fig. 211, line-segment *1-2*, of length $2R$ is the major axis of the horizontal projection of the circle, while line-segment *3'-4'*, equal to $2R$, is the major axis of the vertical projection.

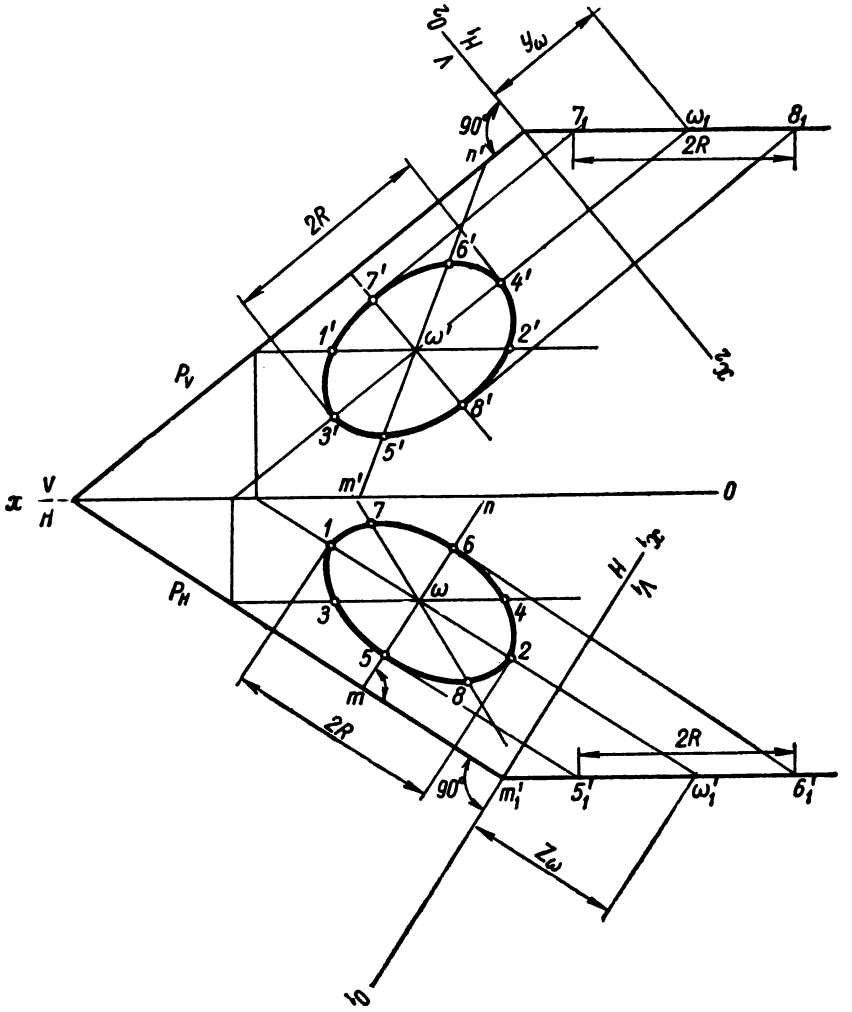


Fig. 211

The minor axes will be obtained when projecting onto the H and V planes those diameters of the circle that (in space) are perpendicular to the horizontal line and vertical line, respectively, of the P plane. This is due to the fact that according to the theorem of projecting a right angle, the projections of only these two diameters will make an angle of 90° with the major axes of the ellipses.

Construction of the minor axis *5-6* in the horizontal projection of the circle is the solution of the following problem: lay off a line-segment (equal to the given R) on the given line MN from a point Ω given on this line.

In the case under consideration, the line MN is the steepest line in the P plane passing through point Ω. This line should contain points *5* and *6*, which are a distance R from Ω. The problem is solved by the method of changing the planes of projection.

Similarly, we find points *7*, *7'* and *8*, *8'* which determine the minor axis of the ellipse in the vertical projection of the circle.

Sec. 39. The Helix

1. Cylindrical helix. *This line is generated by a point in motion along the generating line of a right circular cylinder rotating about its axis so that the path traversed by the point along the generatrix is all the time proportional to the angle of turn of the cylinder.*

Displacement of the point along the generating line during one revolution of the cylinder is called the pitch (or lead) of the cylindrical helix. We also distinguish between right-hand (Fig. 212*a*) and left-hand (Fig. 212*b*) helices.

Fig. 213 shows the construction of the projections of a cylindrical helix. The initial data are: R the radius of the cylinder, h the pitch (or lead) of the helix, and the hand of the helix (right-hand in our case).

Since the angular displacement of a point moving along a cylindrical helix is proportional to the linear displacement, a point turning $\frac{360^\circ}{n}$ should move parallel to the axis of the cylinder by $\frac{1}{n}$ of the lead. In our case $n=12$, and therefore both the circle (which is the horizontal projection of the cylinder) and its altitude, equal to the lead h, are divided into 12 equal divisions.

The vertical projection of each subsequent point (say *2'*) will be displaced relative to the preceding point *1'* along the axis of the cylinder by 1/12 of h; what is more, the projection must lie on the common perpendicular to the x-axis with the corresponding horizontal projection, point *2*.

The vertical projection of a helix is a deformed sine curve because its construction follows that of a sine curve.

Upon development of the cylindrical surface on a plane, the helix develops into a straight line. This is due to the fact that the linear and angular displacement of the point are directly proportional to one another. Consequently, the helix is the geodetic line* of a cylindrical surface.

* A geodetic line (geodesic) is the shortest line between two points on a surface.

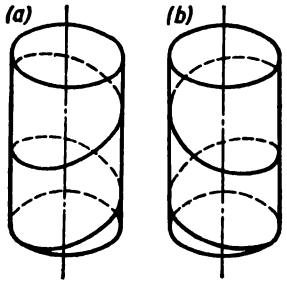


Fig. 212

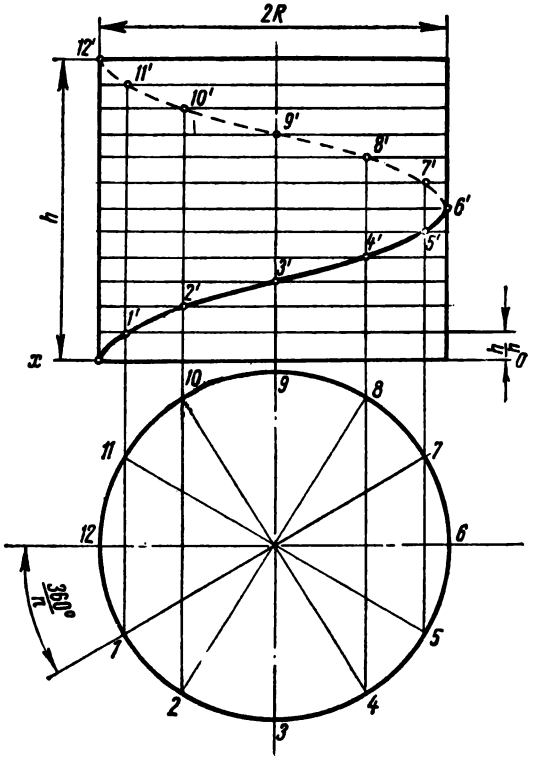


Fig. 213

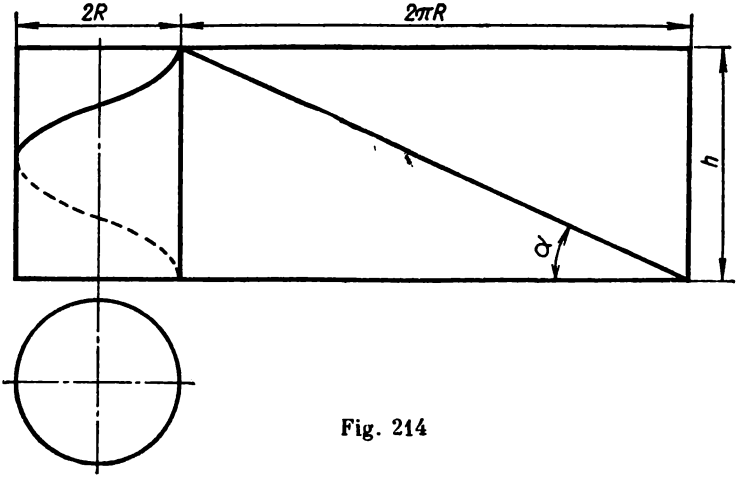


Fig. 214

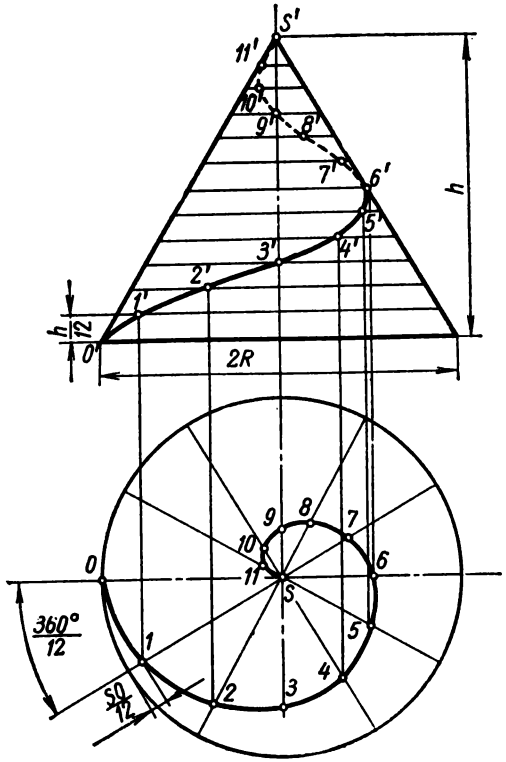


Fig. 215

When considering the development of a cylinder containing a cylindrical helix (Fig. 214), we can establish the relationship between the radius of the cylinder R, the lead h and the helix angle α: $h = 2\pi R \tan \alpha$.

2. **Conical helix.** *This line is generated by a point moving along the generating line of a right circular cone rotating at the same time about its axis so that the path traversed by the point on the generatrix is all the time proportional to the angle of turn of the cone.*

The projection, on the cone axis, of the displacement of a point along the generatrix during one revolution is called the lead (pitch) of the conical helix.

A peculiarity in the construction of the horizontal projection of the conical helix (Fig. 215) lies in the fact that the horizontal projection of a moving point is determined via two motions: rotational (together with the generatrix) and translational (along the generatrix).

Thus, when constructing point *1*, the horizontal projection of the generatrix of cone *SO* was turned in plan through 360°/12, and point *O* was translated along it by 1/12 of the length of *SO*. The remaining points were constructed in the same sequence.

The horizontal projection of a conical helix is the *spiral of Archimedes*. The vertical projection of each point of the helix is determined by the intersection of the vertical projections of parallel lines* of the cone, the planes of which are displaced to a separation of $\frac{h}{12}$, and the lines of recall.

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* Parallel lines of the surface of a cone are circles—lines along which the cone is cut by planes perpendicular to its axis.

CHAPTER VII

CURVED SURFACES

In analytic geometry, a surface given by an equation in current coordinates is regarded as a locus whose coordinates satisfy the given equation:

$$F(x, y, z) = 0$$

When depicting a surface, it is more convenient to regard it as a locus not of points but of certain lines that satisfy definite conditions. These lines, which may be straight or curved, are called generating lines (generatrices). To construct the projections of a surface one must know the type of generatrix and also its mode of motion.

The same surface may be generated in a number of ways. For instance, the surface of a right circular cone may be generated by the revolution of a straight-line generatrix about the axis intersecting it or by the translational motion of a constantly deforming circle, the centre of which moves along the axis of the cone, while the plane of the circle is perpendicular to the axis. Of the great diversity of methods of generating surfaces, we must select those that combine a simple shaped generating line with uncomplicated kinematics.

In appearance we distinguish between *ruled* and *double-curved* surfaces. The generating line of the former is a straight line, that of the latter is a curved line. Ruled surfaces are divided into so-called *developable* surfaces which may be developed onto a plane without folding or rupturing, and *nondevelopable* (*warped*) surfaces.

Classifying surfaces as to the law of motion of the generating line, we find most engineering surfaces divisible into: 1) surfaces of revolution, 2) helical surfaces, and 3) surfaces with plane directors.

In a special category are those curved surfaces which are not generated in accord with any kinematic law. The optimal shapes of such surfaces are determined by the physical conditions under which parts with such surfaces operate and are established in numerous experiments. Illustrations of specially shaped surfaces are turbine and supercharger blades, the fairings of ship hulls, aircraft skin

and automobile body sheeting. These surfaces are studied in special courses for the development of manufacturing processes.

We now consider surfaces of revolution, developable surfaces, helical surfaces, and a surface with a plane director.

Sec. 40. Surfaces of Revolution

A surface of revolution is generated by the revolution of a curved-line or straight-line generatrix about a fixed axis.

Fig. 216 shows a surface generated by revolution about axis II_1 of the curved line ABC lying in the plane passing through the axis.

As the generatrix revolves, each point of it moves in a circle, the plane of which is perpendicular to the axis of the surface; these circles, as has already been pointed out, are called parallels, the smallest one being what is known as the gorge circle.

Two basic properties follow from the law of generation of surfaces of revolution:

(1) a plane perpendicular to the axis of revolution intersects the surface in a circle (a right section, or parallel);

(2) a plane passing through the axis of revolution intersects the surface in two lines symmetric about the axis called meridians.

In drawings, the axis of a surface of revolution is ordinarily made perpendicular

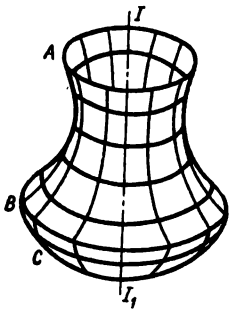


Fig. 216

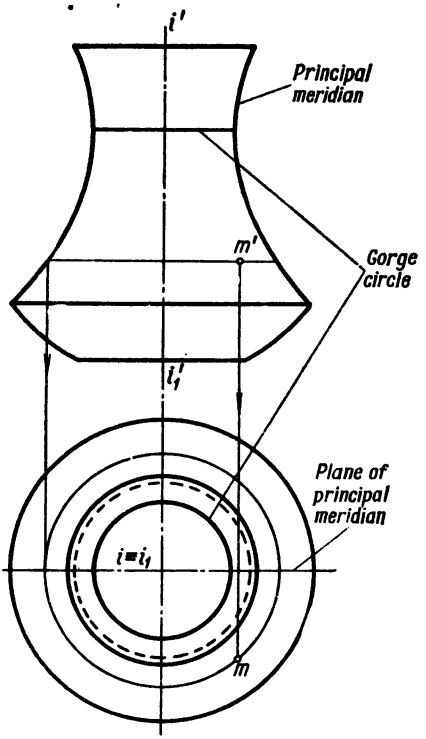


Fig. 217

to one of the projection planes. Thus, in Fig. 217 the axis II_1 is perpendicular to H.

In this case, all parallel lines project true size on the H plane, and the two meridian lines that determine the vertical outline of the surface project true size on the V plane. The meridian line in the plane parallel to V is called the principal meridian.

To find the horizontal projection of an arbitrary point M lying in the surface of revolution, draw through m' the vertical projection of the parallel. Then determine m after constructing the projection of this parallel on the H plane. Fig. 217 shows point M, which is visible from the front.

We now consider the most common surfaces of revolution with curved-line generatrices.

1. The *sphere* is generated by revolving a circle about a diameter.

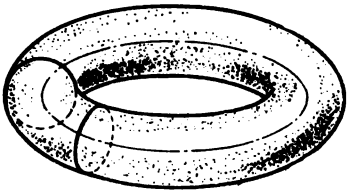


Fig. 218

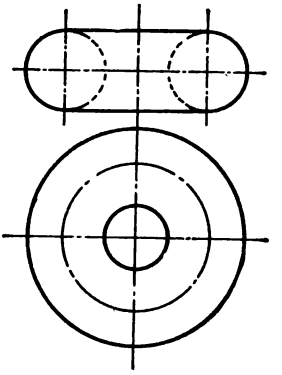


Fig. 219

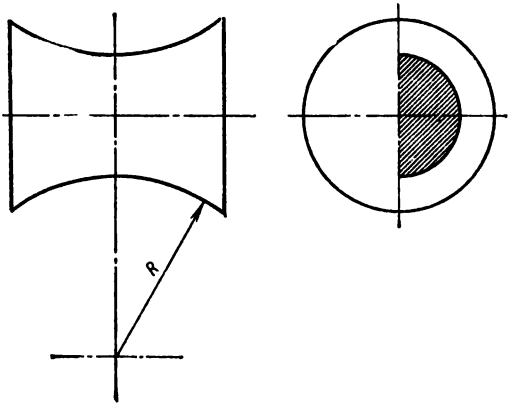


Fig. 220

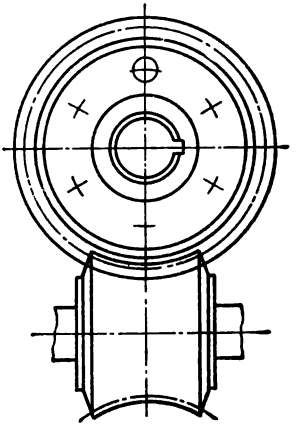


Fig. 221

2. The *torus* is generated by revolution of a circle lying in the plane of the axis but not passing through the centre of the circle. A torus whose axis does not intersect the circle is often called an annular torus (Fig. 218). The orthogonal projections of a torus are shown in Fig. 219.

3. The *globoid*. The generating line of this surface is an arc of a circle whose plane may, in the general case, not coincide with the axis of revolution (Fig. 220).

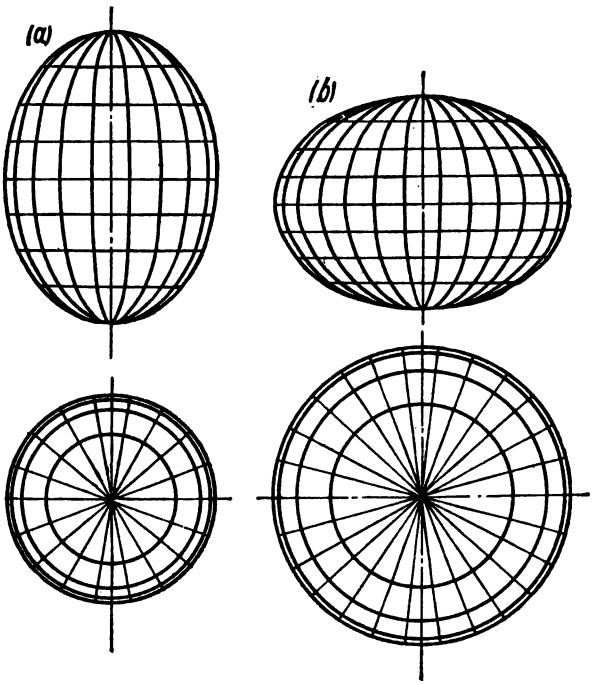


Fig. 222

In gear designing, globoidal shapes are used for worm gears and the wheels of Hindley worm gears (Fig. 221), which are superior to cylindrical worm gears in efficiency and compactness.

4. An *ellipsoid of revolution* is generated by the revolution of an ellipse about one of its axes (major or minor). If the major axis is the axis of revolution, the ellipsoid of revolution is elongated (Fig. 222*a*), if it is the minor axis, we have an oblate ellipsoid or *spheroid* (Fig. 222*b*). The earth (*geoid*) closely resembles a spheroid.

5. The *paraboloid of revolution* is generated by revolving a parabola about its axis.

This is the surface of parabolic reflectors used in searchlights (Fig. 223) and automobile headlights, all of which utilize the focal

Fig. 223

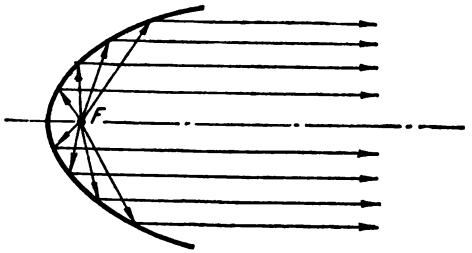


Fig. 224

property of the parabola: a light source placed in the focus of a parabola sends out rays reflected in parallel lines (Fig. 224). This property also underlies sound detecting devices and radio telescopes.

6. The *hyperboloid of revolution.* We distinguish the hyperboloid of revolution of one nappe and of two nappes. The first surface is obtained by revolving a hyperbola about its imaginary axis (Fig. 225), the second surface by revolving a hyperbola about its real axis (Fig. 226).

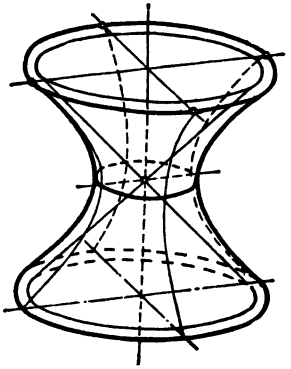


Fig. 225

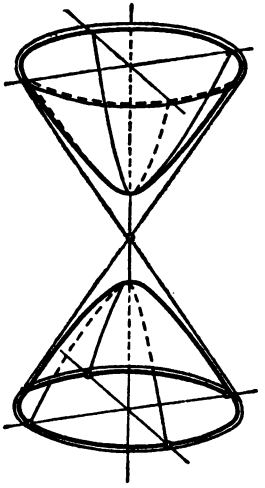


Fig. 226

As will be shown below, the surface of a hyperboloid of one nappe may also be generated by revolving a straight line. This is a double-ruled surface, which means that through each point of a hyperboloid of revolution of one nappe there pass two and only two straight-line generatrices.

Sec. 41. Certain Properties of a Hyperboloid of Revolution of One Nappe and Its Application in Building Construction

We consider the hyperboloid of revolution of one nappe, which is a general type of ruled surface of revolution.

Let there be given two skew lines (Fig. 227) AB and II_1, the former being the generatrix of a surface of revolution, the latter (II_1)

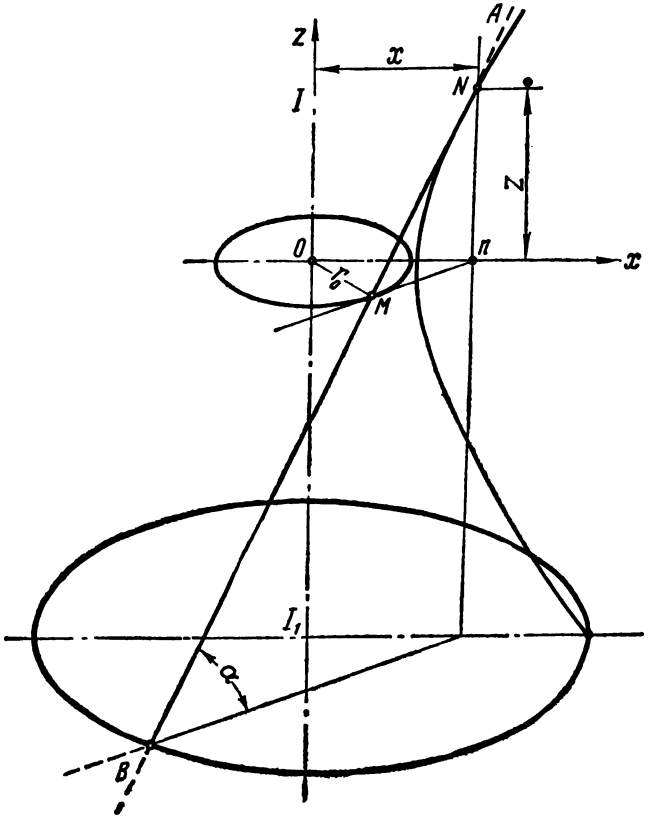


Fig. 227

the axis. As the straight line AB is revolved about II_1, each point of the generating line will move in a circular path whose plane is perpendicular to axis II_1.

Obviously, one of the parallels is a circle of minimal radius r_0 equal to the shortest distance between the given straight lines (seg-

ment OM). This circle is called the circle of the gorge. We shall demonstrate that the meridian line of such a surface is a hyperbola, in other words that this surface is a hyperboloid of revolution of one nappe.

Theorem 1. The meridian of a surface generated by revolving one skew line about another is a hyperbola.

We shall determine the equation of the meridian line in a fixed system of coordinates Oxz located so that the origin coincides with the centre of the gorge circle and the z-axis with the axis of the hyperboloid II_1 (see Fig. 227).

Now let a straight-line generatrix intersect the plane of the coordinate axes in point N lying in the meridian plane.

To establish the relationship between the coordinates x and z of point N consider the right triangle OMn (OM is perpendicular to Mn), in which

$$On^2 = OM^2 + Mn^2$$

and since On is the abscissa of point N, segment OM is the radius r_0 of the gorge circle and $Mn = \frac{z}{\tan \alpha}$, we have

$$x^2 = r_0^2 + \frac{z^2}{\tan^2 \alpha}$$

and, finally, $\frac{x^2}{r_0^2} - \frac{z^2}{r_0^2 \tan^2 \alpha} = 1$. Here α is the angle between the generatrix AB and a plane perpendicular to the axis of revolution; z is a coordinate of point N.

We thus get an equation of the form $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$, which is the equation of the hyperbola.

A hyperboloid of revolution of one nappe may be regarded as a general-type ruled surface of revolution that degenerates, in special cases, into a conical and cylindrical surface.

Thus, the surface of a hyperboloid of revolution degenerates into a cone when $r_0 = 0$ and $\alpha \neq 0°$, into a cylinder when $r_0 \neq 0$ and $\alpha = 90°$, into a plane with a hole when $r_0 \neq 0$ and $\alpha = 0°$ and into a straight line when $r_0 = 0$ and $\alpha = 90°$.

We consider certain properties of a hyperboloid of revolution of one nappe.

In Fig. 228, line AB revolving about axis II_1 occupies successive positions A_1B_1, A_2B_2, etc., to form a family of straight-line generatrices.

It may be shown that the surface of one and the same hyperboloid of revolution of one nappe contains not one but two families of straight-line generatrices.

Theorem 2. Two and only two straight-line generatrices pass through each point of a hyperboloid of revolution of one nappe.

Indeed, let point M (Fig. 229) be on the surface of a hyperboloid of one nappe whose axis is the line II_1 and whose generating line is AB. The position of AB of the first family of generatrices is determined by point M, the distance of the line from axis II_1 (equal to r_0),

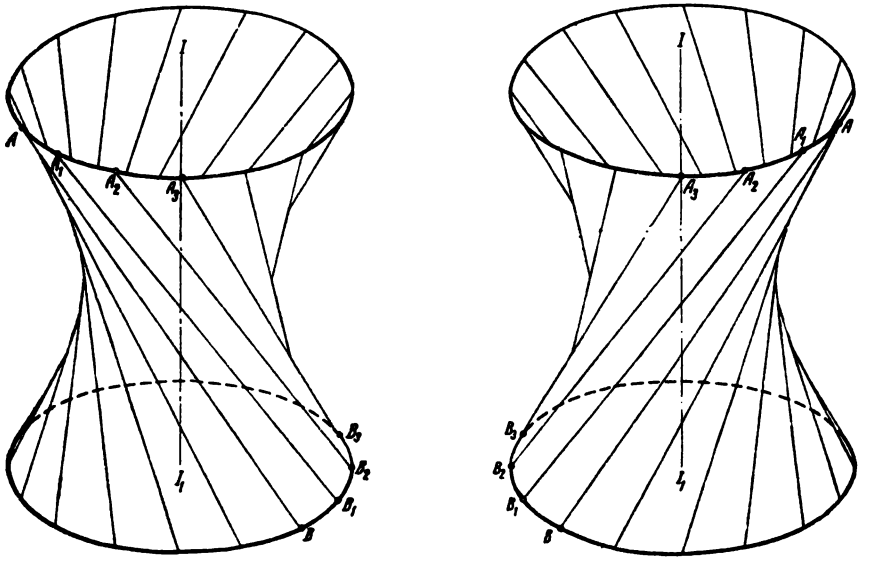


Fig. 228

and angle α, which AB makes with a plane perpendicular to II_1. But these same conditions will be satisfied by yet another straight line CD.

Each pair of points of lines AB and CD located in a single plane perpendicular to axis II_1 (say points K and K_1) will be situated at the same distance from the axis, as follows from the equality of the triangles KI_1m and K_1I_1m. Hence, as lines AB and CD revolve, both points will move in the same circular path (parallel). The line CD, revolving about axis II_1, will generate the same hyperboloid of revolution of one nappe as line AB. Successive positions of CD form a second family of straight-line generatrices of the same hyperboloid.

Each generatrix of one family intersects all the generatrices of the other, except one, which is parallel to it.

All straight-line generatrices of one family are skew lines.

The proofs of these properties of the generatrices of a hyperboloid are given in courses of analytic geometry.

Fig. 230 illustrates the construction of the projections of a hyperboloid of revolution of one nappe. Let the axis of revolution (straight line II_1) be perpendicular to the H plane. As generatrix AB revolves about axis II_1, each point of the line will move in a circular path in space (a parallel), the plane of which is perpendicular to axis II_1. Thus, this circle will project on the H plane true size, and on the V

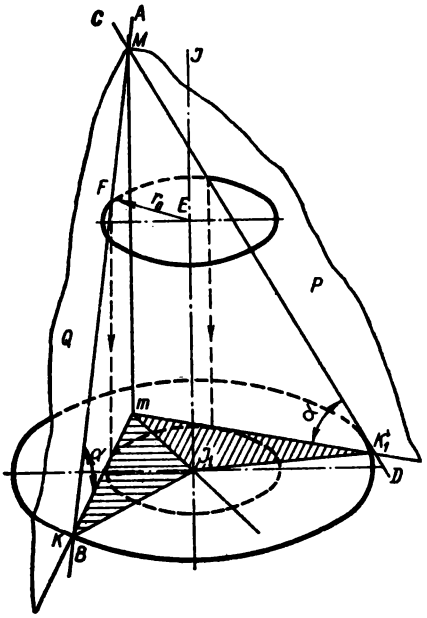


Fig. 229

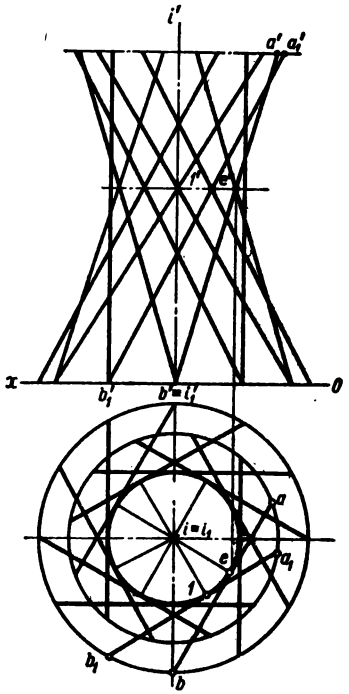
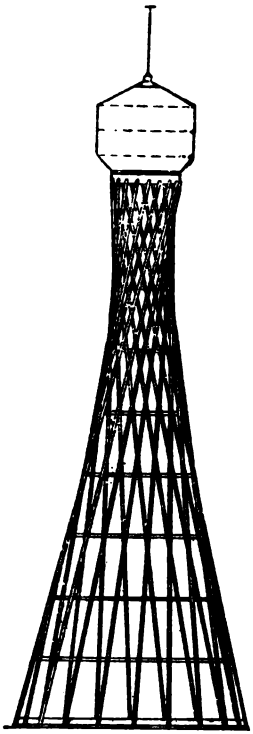


Fig. 230

plane into a straight line parallel to the x-axis. The point (E) of the generatrix closest to the axis of revolution will describe a circle of minimal radius, the gorge circle. Since the horizontal projections of all generating lines must be tangent to the projection of the gorge circle, each successive position of the straight-line generatrix may be established by drawing tangent lines to the projection of the gorge circle. In Fig. 230, this circle is divided into twelve equal parts. The tangent line a_1b_1 (the horizontal projection of a generatrix turned through 30°) is drawn to the projection of the circle in point I.



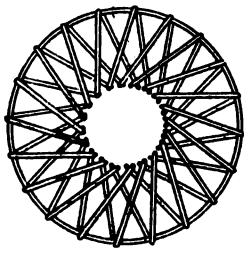


Fig. 231

Its vertical projection is determined by points a_1' and b_1', each of which lies in the plane of its parallel circle. The remaining generatrices are constructed in a similar fashion.

Credit for applying the hyperboloid of revolution of one nappe to building structures goes to the prominent Russian engineer and honorary member of the USSR Academy of Sciences V. G. Shukhov (1853-1939). In 1896, Shukhov was granted a so-called "Privilege" for a "lattice-work tower featuring a frame that consists of intersecting straight-line wooden rods (or metallic pipes, or elbows) positioned along the generatrices of a solid of revolution in the shape of a tower, which rods (etc.) are riveted together at points of intersection and are, in addition, held in position by horizontal rings" (Fig. 231). In that same year of 1896, Shukhov built a lattice-work tower at the Nizhegorodsky Exhibition that served as a supporting structure for a reservoir with a capacity of 10,000 pails of water.

Technological elegance (no bending or shaping of elements was required) was combined with beautiful architectural forms, rational strength, stability and economy of metal. These advantages in design were widely utilized in the construction of radio masts, beacons, water towers, and pylons for electric power transmission lines. In 1921, Shukhov designed and supervised the construction, in Moscow, of a 160-metre radio tower made up of six sections (six hyperboloids) placed one on the other. This over the years has been the emblem of Russian radio engineering and is at the same time a monument to the genius of Shukhov, whose fruitful work serves as a brilliant example of creative initiative and persistence in bringing ideas to fruition.

Sec. 42. Developable Surfaces

A developable surface is a ruled surface that may be developed on a surface without folding or tearing. The fact that a surface is ruled is a necessary but not sufficient condition for developability.

Differential geometry has a proof that any developable surface will be either a cylindrical or conic surface or the locus of tangent lines to a certain curve.

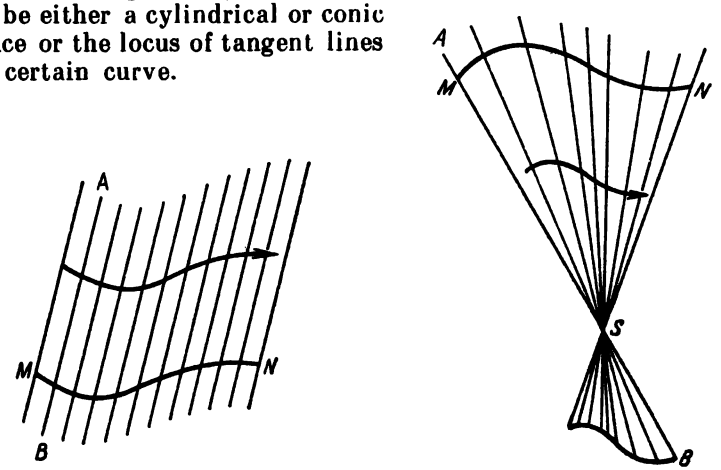


Fig. 232

The construction of cylindrical and conic surfaces was considered earlier, when we discussed the generation of prismatic and pyramidal surfaces.

Fig. 232 illustrates methods of the generation of cylindrical and conic surfaces. The generating line AB of the first surface moves, remaining parallel to itself, along the directrix MN, while the generating line of the second surface moves along the directrix and passes through a fixed point S.

Fig. 233 depicts a surface that is the locus of lines tangent to an arbitrary space curve MN.

Curve MN is called a *cuspidal edge*. When the point of tangency of a straight-line generatrix moves along a cuspidal edge, a line tangent to it forms a developable surface, and the half-tangent that lies to one side of the point of tangency forms one nappe, while the other half-tangent forms the other nappe. In this way the cuspidal edge divides the surface into two nappes.

We get a pictorial representation of the peculiar surface structure near a cuspidal edge by cutting it with a plane that is perpendicular to the generatrix and passes through the point of tangency of the given generatrix.

Fig. 233 shows such a section: a curve of the type M_1KN_1, which has a cusp at K. The branches of the curve, M_1K and N_1K, meet at this point. The points of the cuspidal edge are singular points of a developable surface.

The projections of a surface with a cuspidal edge are shown in Fig. 234. Here, the edge of the surface is curve MN, to which are drawn the tangent lines MM_1, AA_1, EF, BB_1, and NN_1. Plane Q is passed perpendicular to EF through the point of tangency of line

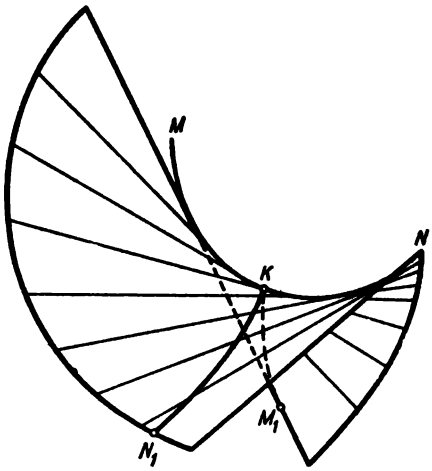


Fig. 233

EF and the cuspidal edge (point K). And since the straight line is in a horizontal position, plane Q will be a horizontal projecting plane. Fig. 234 shows the projections of points M_1, A_1, K, B_1, and N_1, in which the straight-line generatrices intersect plane Q.

The locus of these points is a section of the surface by plane Q. Point K will be a singular point of the curve.

In the given case, this rough idea of the structure of the surface enables us to construct the vertical projection of the section: m_1'-a_1'-k'-b_1'-n_1.

Developable surfaces possess a valuable technological property: they may be worked with plane tools whose motion is determined by a single parameter.

During the working operation, the plane of the cutting tool moves relative to the workpiece forming a family of planes of one parameter. The parameter may be time, the angular displacement of the tool, and so forth. The worked surface will consequently be the envelope of a one-parameter family of planes, which (as is proved

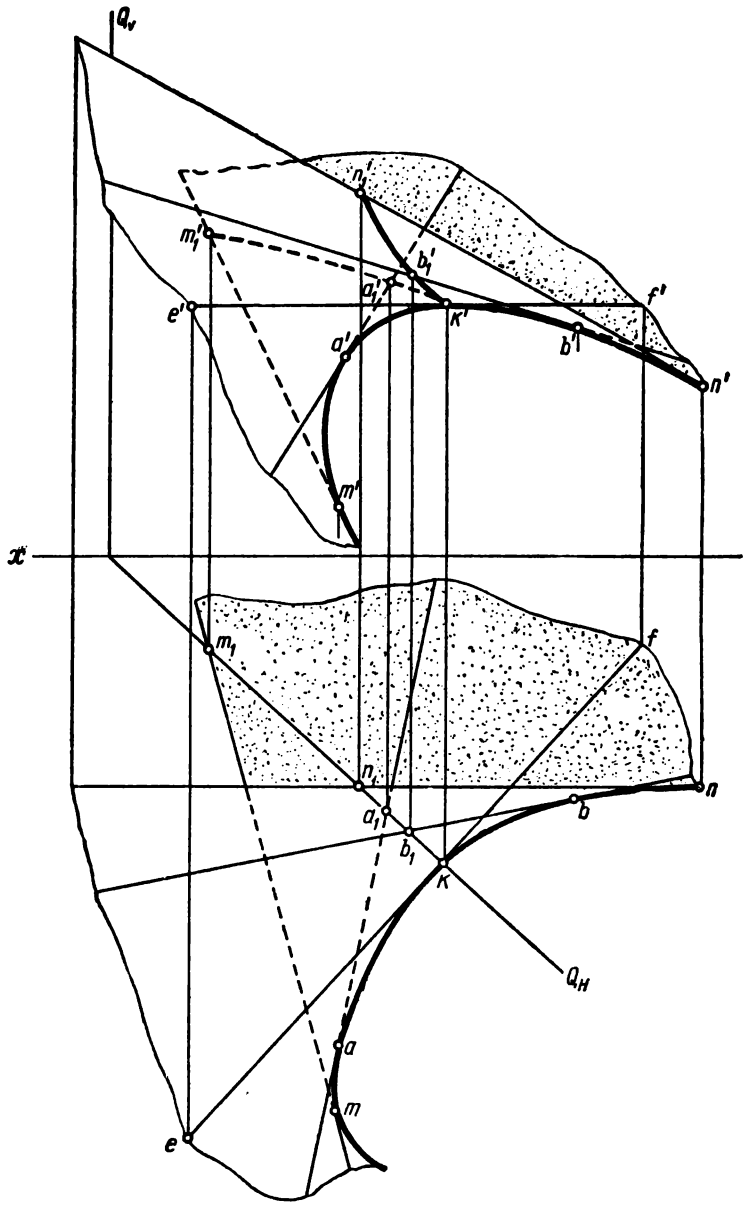


Fig. 234

in differential geometry) is necessary and sufficient for the surface to be developable.

We note an important case of a developable surface when a *cylindrical helix serves as the cuspidal edge of the surface*. This surface is of interest not only due to its special geometric properties but also because of its technical applications.

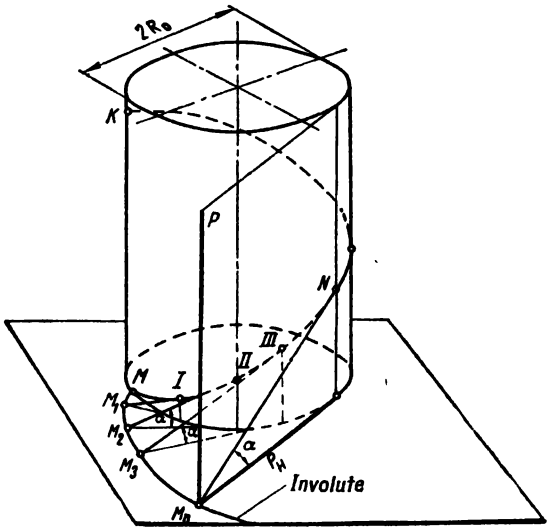


Fig. 235

Let the cylindrical helix MNK (Fig. 235) be the cuspidal edge. The figure also shows lines tangent to the helix (the straight lines M_1I, M_2II, M_3III, ..., the locus of which generates a developable surface).

It is readily shown that each tangent line makes with a plane perpendicular to the cylinder axis an angle α (equal to the angle of rise of the helix).

Indeed, let us consider the development of the surface of a cylinder on a plane P, when this plane remains tangent to the cylinder.

The straight line obtained in the development of the helix will consecutively coincide with lines tangent to the helix.

Consequently, all lines tangent to the cuspidal edge make with a plane perpendicular to the cylinder axis a constant angle equal to the slope angle of the straight line that is produced by development of the helix. Now this angle is also equal to the angle of rise of the helix on the cylinder.

A surface, all straight-line generatrices of which make the same angle with a certain plane, is called a *surface of constant slope.*

One of the special cases of such a surface is the developable surface under consideration.

If a surface, the cuspidal edge of which is a cylindrical helix, is cut by a plane perpendicular to the cylinder axis, then the section yields

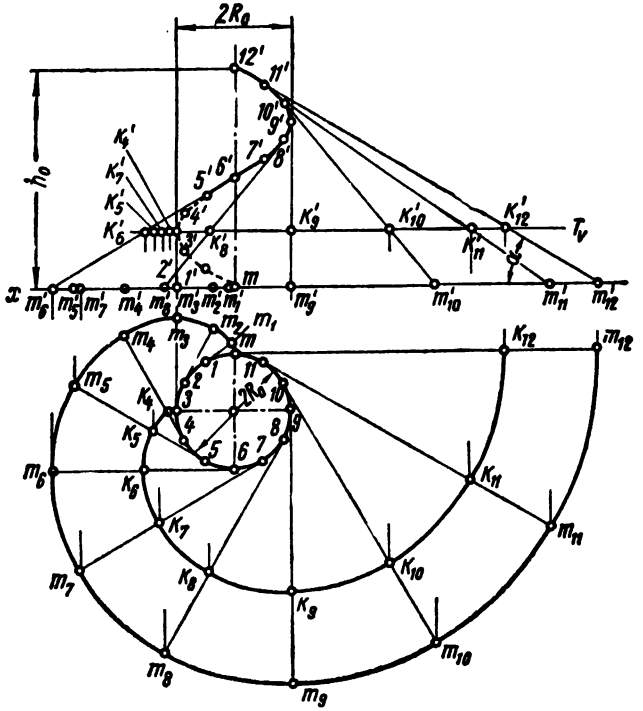


Fig. 236

an *involute*, the evolute * of which is a circle—the orthogonal projection of the cuspidal edge on this same plane.

The truth of this assertion stems from the fact that in the generation of the surface (when the straight line is in motion in space remaining tangent to the cylindrical helix) the projection of the tangent line on a plane perpendicular to the cylinder axis rolls without sliding round a circle that is the projection of the cuspidal edge on

* The evolute of a curved line is the locus of its centres of curvature. An involute of a given curve is any curve whose evolute is the given curve. The normals of an involute are tangent to the evolute.

this plane. When a straight line moves in this fashion round a circle, each point of the line describes an involute.

For this reason, this surface is frequently called an *involute screw* surface.

Fig. 236 portrays the projections of such a surface where the straight lines M_1-I, M_2-II, M_3-III, etc., are tangent to the helix M-I-II-III, etc., constructed on a cylinder of radius R_0. This is a right-hand cylindrical helix with a lead h_0. The locus of the horizontal traces of lines tangent to the cylindrical helix is the involute of the circle of radius R_0. The points of the involute are designated by M, M_1, M_2, etc. A section of this surface by a plane T, perpendicular to the axis of the cylinder, will also be an involute, each point of which, K_4, K_5, K_6, etc., is defined as the intersection point of the appropriate tangent with the plane T (T is parallel to H). The involute screw surface cuts the cylinder coaxial with it in cylindrical helices having leads equal to the lead of the cuspidal edge of the surface.

Indeed, when the point of tangency of a straight-line generatrix (in motion along a cuspidal edge) is translated in the direction of the cylinder axis by the amount of the lead h_0, for a constant angle α (α is the angle of slope of the generatrices to a plane perpendicular to the axis of the surface), any other point of the same generatrix will be displaced parallel to the axis of the cylinder by the lead h_0.

The embankments of fills and cuts of an earthen railway bed on an upgrade curve are involute screw surfaces.

To establish a relationship between the geometric parameters of the cuspidal edge and the embankment surface of a road, we refer to Fig. 237.

Here we have the projections of the road bed, the outer edge of which (the curve A_0-A_1-A_2, etc.) is a helix. Note that the axis of the road and the inner edge are also cylindrical helices, but with a different angle of rise. *

The horizontal projection of the outer edge is an arc of a circle of radius R_1 drawn from the centre ω.

Taking advantage of the relation $\tan \alpha_1 = I_1$, we can determine the angle of rise of the edge α_1 from the given gradient of the road I_1. Noting on the horizontal projection of the outer edge a number of points a_0, a_1, a_2, etc., which are displaced along the arc relative to one another by the amount $R_1\varphi$ (φ is the central angle in radians), we construct their vertical projections a_0', a_1', a_2', etc. The vertical displacement of each of the points will be determined by the quantity

$$\Delta z = R_1 \varphi \tan \alpha_1$$

* For considerable R_1 and small road width l, the slopes of the edges and the axis are considered equal.

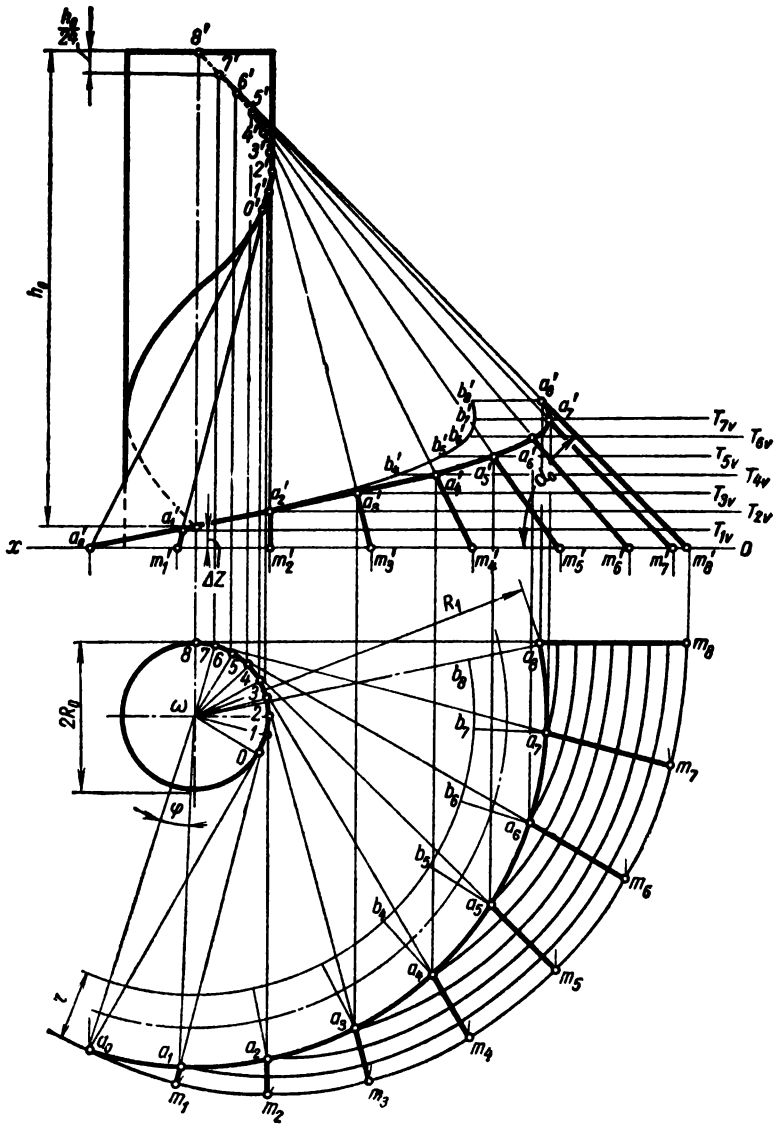


Fig. 237

Besides the parameters R_1 and α_1, we also make use of the amount of incline of the embankment i to determine the geometry of the surface of the embankment. The surface slope of the embankment $i=\tan\alpha_0$, where α_0 is the angle of the straight-line generatrix with the horizontal plane.

It is assumed that these parameters are given. The problem consists in determining the lead of the cuspidal edge and the radius of the cylinder containing the edge.

The angle of rise of the cuspidal edge must be equal to the angle of inclination of the generatrices of the slope surface (angle α_0). Since the desired cuspidal edge and outer edge of the road must lie in the same involute screw surface,

$$h_0 = h_1$$

where h_0 is the lead of the cuspidal edge, and h_1 is the lead of the road edge.

From the foregoing we can find the radius R_0 of the cylinder whose surface contains the cuspidal edge.

Indeed, $h_0=2\pi R_0\tan\alpha_0$ and $h_1=2\pi R_1\tan\alpha_1$.

Hence,

$$R_0\tan\alpha_0 = R_1\tan\alpha_1,$$

whence

$$R_0 = R_1\frac{\tan\alpha_1}{\tan\alpha_0}$$

or

$$R_0 = R_1\frac{I_1}{i}$$

Thus, the quantity R_0 is determined by three independent parameters R_1, I_1, and i, which characterize the geometry of the surface of a road embankment on a rising curve.

Constructing the projection of a cylinder of radius R_0 that is coaxial with the road edge (the cylinder axis passes through point ω), we draw the horizontal projections of tangents to the cuspidal edge: the straight lines a_0-*0*, a_1-*1*, a_2-*2*, etc. Points *0*, *1*, *2*, etc., belong to the horizontal projection of the desired cuspidal edge. The vertical projection of one of them, namely *8′*, is readily obtained by noting that the tangent line A_8-*VIII* is parallel to the V plane because in the orthographic representation (a_8-*8*) is parallel to the x-axis. The angle between the vertical projection a_8'-*8′* of this tangent line and the x-axis must be equal to the given angle α_0, which is the angle of inclination of the generatrices of the surface of the embankment to the horizontal plane. The intersection of the straight line passing through point a_8' at an angle α_0 to the x-axis, and the line of recall yields point *8′*.

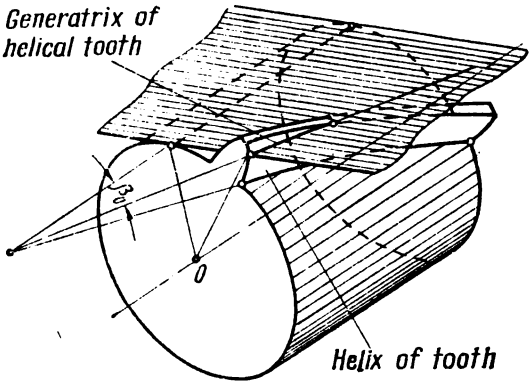


Fig. 238

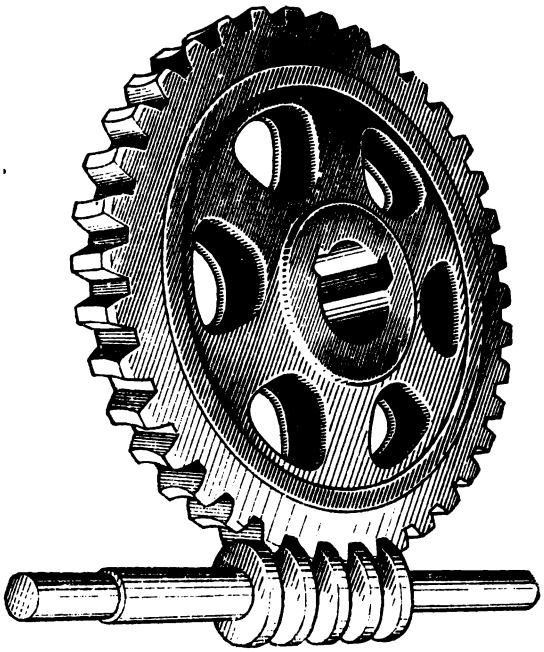


Fig. 239

Knowing the amount of lead h_0 and the central angle $\varphi=15°$, which determines the angular displacement of each subsequent point relative to the preceding one, we find the vertical projections of the remaining points of the cuspidal edge, and in the sequence first $7'$, then $6'$, and so forth.

Obviously, when $\varphi=\frac{1}{24}360°$, the lead h_0 has to be divided into 24 parts as well.

Connecting the points found, $0'$, $1'$, $2'$, etc., with the respective points a_0', a_1', a_2', etc., lying on the outer road edge, we get the vertical projections of the straight-line generatrices of the involute screw surface. Those that belong to the surface of the fill embankment are depicted somewhat heavier. The horizontal projection plane H, and the planes T_1, T_2, T_3, etc., parallel to it intersect the surface of the embankment along curves that are involutes of a circle of radius R_0. Their construction does not in any way differ from those in Fig. 236, where we determine the projections of sections of the involute screw surface by the planes H and T.

Other applications of this surface include the lateral surfaces of skew teeth of cylindrical gears (illustrated in Fig. 238), the working surfaces of involute worm gears (Fig. 239), the surfaces of the cutting edges of gear cutters used to cut the teeth of gears and wheel screws.

Sec. 43. Screw Surfaces

A screw surface is generated by the helical motion of some line, which is called the generatrix of the surface.

Helical motion is ordinarily understood to be motion representing a combination of two motions: translational (parallel to a certain axis, the screw axis) and rotational (about this axis). The translational and angular displacements of the line are related as follows:

$$\Delta h = k\,\Delta\varphi$$

where Δh is the linear translation of the generatrix during time Δt, $\Delta\varphi$ is its angular displacement during the same time, and k is a proportionality factor.

If k is constant, we have a surface with a constant lead;* if k is variable, the lead of the surface is also variable.

If the helical motion is performed by a straight line, the surface will be a helicoid. Depending on how the straight-line generatrix of the screw surface is located relative to the screw axis, we distinguish the following screw surfaces: *Archimedean, involute, and convolute.*

* The lead of a screw surface is the linear displacement of the generatrix during one revolution.

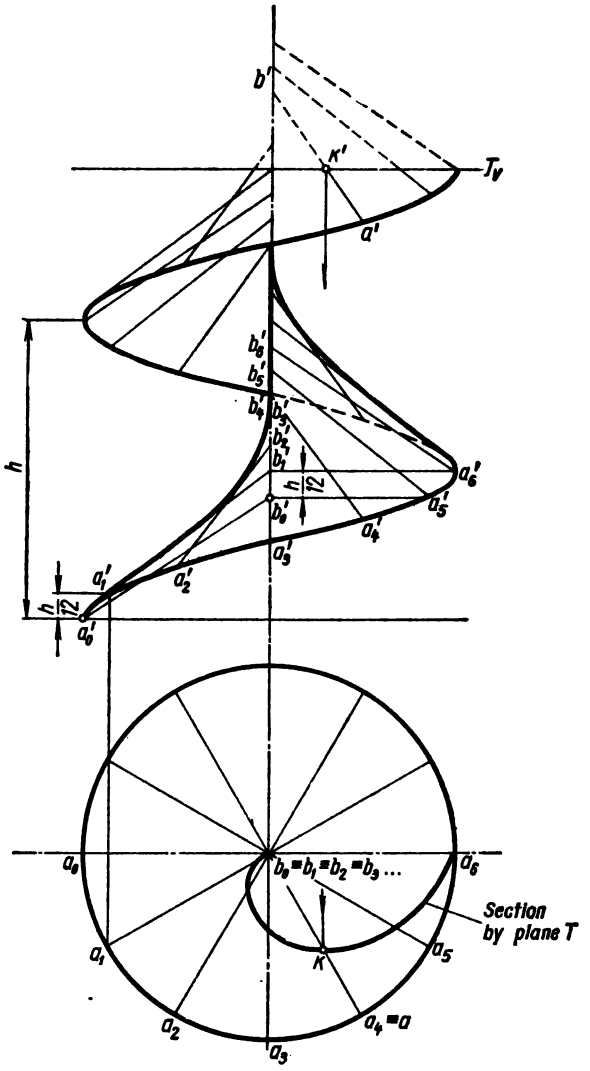


Fig. 240

The Archimedean screw surface is obtained by the helical motion of a straight line A_0B_0 intersecting the screw axis (Fig. 240).

All points of the straight-line generatrix (except one) will move along cylindrical helices. Fig. 240 shows a helix described by point A_0. One of the points of the generatrix (that in which the generatrix

intersects the axis of the surface) will not participate in the helical motion. This point, B_0, only performs rectilinear motion along the surface axis; when the generatrix turns through an angle $\varphi = \frac{360°}{n}$, point B_0, together with the straight line, is displaced along the axis by the amount of $\frac{1}{n}$ part of the lead h.

Joining the points of the helix A_1, A_2, A_3, etc., with the respective points B_1, B_2, B_3, etc., situated on the axis, we get straight lines, the locus of which represents an Archimedean screw surface.

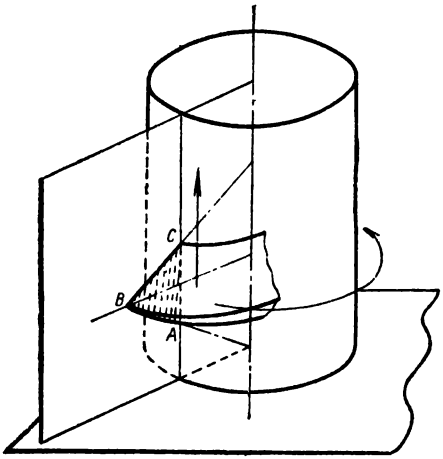


Fig. 241

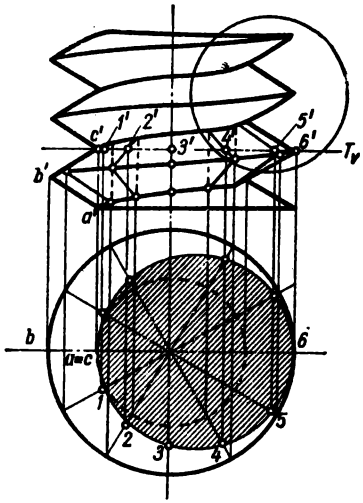


Fig. 242

The section of this surface by plane T perpendicular to its axis yields the spiral of Archimedes. This surface (the Archimedean screw) is used in various machines for conveying loose materials, for mixing viscous fluids, in screw conveyors, mixing devices, and the like.

Archimedean screw surfaces are found in worm cutters used to cut gear wheels and the working surfaces of worms in worm-gear drives. Threaded items have Archimedean surfaces. Fig. 241 gives a schematic view of the cutting of a V-thread. In profile, this thread is an equilateral triangle ABC with its plane passing through the axis of the screw. In the helical motion of the profile over the cylinder, the two sides AB and BC of the triangle generate Archimedean surfaces.

The vertices of the triangular profile ABC move along helical lines whose projections are constructed in Fig. 242. The visible contour of the screw in the vertical projection is bounded by common tangent lines to the outer and inner helices (this is shown magnified in Fig. 243).

The section of the screw by a horizontal plane T yields a figure bounded by Archimedean spirals (see Fig. 242).

To construct the spirals we had to determine the points of intersection of the sides of triangle ABC (which executes the helical motion) with plane T. The first six points (*I-VI*) are intersection points of plane T and side BC in different positions of the latter.

The upper half of the section will be symmetric with the lower half relative to the horizontal axis. Each point of the contour of the upper half of the section is an intersection point of plane T and side AB (of the triangular profile) that screws into the plane from below.

As plane T is constantly displaced upwards, the figure of the section revolves about the screw axis counterclockwise without changing shape.

If the generatrix intersects the screw axis at a right angle, the helicoid is called right. The right helicoid is used in spiral stairways and ramps (in multistorey garages, for instance). The surface of a railway bed on an upgrade curve (see Fig. 237) is a right helicoid. The straight-line generatrices of the bed surface in Fig. 237 are indicated by A_4B_4, A_5B_5, A_6B_6. When extended, each of them intersects the cylinder axis at 90°. The bed surface in this case is formed by a horizontal straight line executing helical motion.

The involute screw surface considered above (see Figs. 235-237) as a developable surface, which is the locus of tangents to a cylindrical helix, may be generated differently: by the helical motion of a generatrix that does not intersect the screw axis.

The motion of the generatrix occurs as follows: the helix angle of the cylindrical helix described by a point of the generatrix closest to the screw axis must be equal to the slope angle of the generatrix to a plane perpendicular to the screw axis.

A straight-line generatrix performing helical motion will be tangent to a certain cylinder of radius R_0 (see Fig. 235). This radius must be equal to the shortest distance between the axis and the generatrix. The point of tangency N of a straight line M_nN will describe a helical line with helix angle α equal to the slope angle

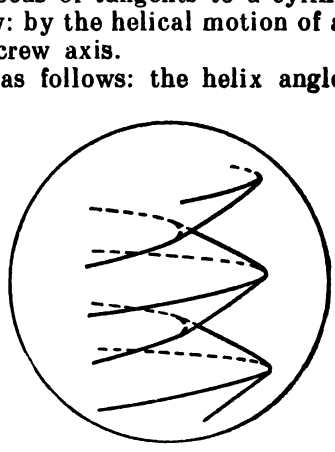


Fig. 243

of the straight line to a plane perpendicular to the screw axis.

Certain properties and applications of an involute screw surface were examined in Sec. 42.

The helical-convolute surface is generated by the helical motion of a straight line whose point of tangency with the basic cylinder

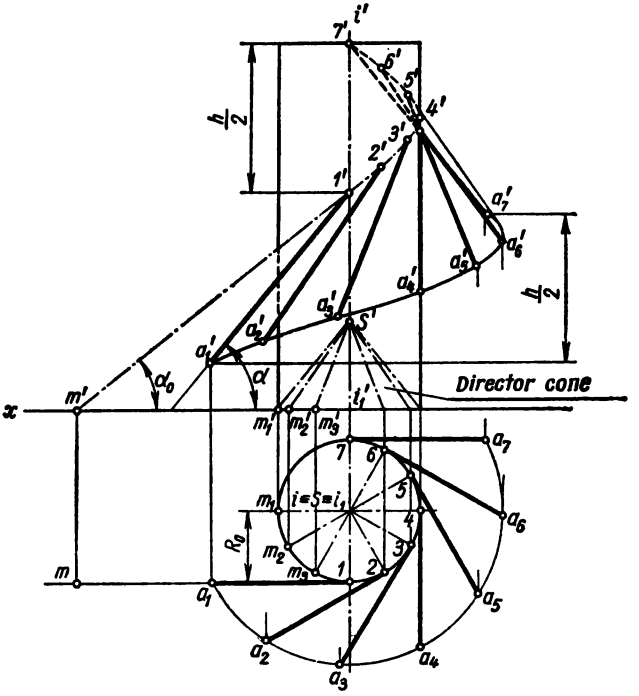


Fig. 244

describes a helix with a rise angle greater or smaller than the angle of slope of the straight line to a plane perpendicular to the screw axis. Thus, the involute screw surface is a special case of the convolute surface. But if the parameters R_0 and α, which describe the relative positions of two skew lines, fully determine the involute screw surface, we need to know yet another parameter: the lead h of the screw surface, which in this case must not be equal to $2\pi R_0 \tan \alpha$ (here, α is the slope angle of the generatrix to a plane perpendicular to the axis of the surface).

But $h=2\pi R_0 \tan \alpha_0$, where α_0 is the rise angle of this same helix along which the point of the generatrix closest to the axis of the surface moves.

And so, given are the axis of the surface—the straight line II_1, the straight-line generatrix A_1-I, and the lead h. The relative positions of the axis II_1 and the line A_1-I determine the segment R_0

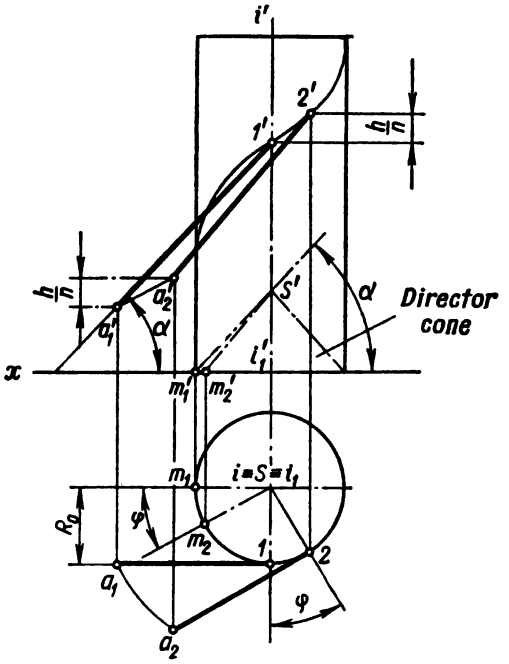


Fig. 245

and angle α_0 (Fig. 244). Since each point of the generatrix executes motion along a cylindrical helix, the projections of the generatrices may be constructed simply by drawing two helices with the same lead h. Fig. 244 shows cylindrical helices that describe points A_1 and I belonging to the generatrix. Joining points a'_2, a'_3, a'_4, ... of the first helix with the respective points $2'$, $3'$, $4'$, ..., lying on the second helix, we get the vertical projections of the generatrices of the convolute surface. Their horizontal projections a_1-1, a_2-2, a_3-3, ... are tangents to a circle of radius R_0, which serves as the horizontal projection of the helix described by point I. Fig. 244 shows the tangent M-I to this helix at point I.

Angle α_0 between the tangent M-I and plane H (H is perpendicular to II_1) is equal to the angle of rise of the helix I-II-III.... In this case $\alpha_0 < \alpha$.

When constructing the projections of screw surfaces it is advisable to make use of what is called a director cone; this cone of revolution is ordinarily positioned coaxially with the screw surface. The cone generatrices should make an angle α with a plane perpendicular to axis II_1. Then the travelling generatrix of a screw surface will remain parallel to the generatrices of the director cone.

Thus, if the straight-line generatrix A-I of a screw surface is turned through an angle φ about axis II_1 (Fig. 245), then in the new position A_2-II it will be parallel to the generatrix of cone SM_2. The horizontal projection sm_2 is drawn parallel to a_2-2. We then determine the vertical projection $s'm'_2$ of the generatrix of the cone, parallel to which the vertical projection of straight line A_2-II will pass through point 2

Using the director cone we are able to dispense with the construction of two helices and need only one. In our case (see Fig. 244) we had only to draw a single helix I-II-III.... Then the vertical projections of the straight-line generatrices are drawn through points I, II, III, etc., parallel, respectively, to $s'm'_1$, $s'm'_2$, $s'm'_3$, etc.

Like the Archimedean and involute surfaces, convolute surfaces are used to produce the working surfaces of worms in gear drives.

Sec. 44. Ruled Surfaces with Plane Director

A plane director is determined by two intersecting straight lines that are respectively parallel to given skew lines. Surfaces with plane directors are loci of straight lines (generatrices) parallel to the plane directors and intersecting two given lines called directrices. If the directrices are two curved lines, the surface is a cylindroid. If one of the directrices is a straight line and the other a curved line, the surface is a conoid, and, finally, if both directrices are straight-line, then the surface is a hyperbolic paraboloid.

The surface of a cylindroid is determined by the plane director P (Fig. 246) and two curved-line directrices KL and MN, which may be space curves or plane curves. In the latter case, the planes of the directrices must not coincide. If the straight line AB remains parallel to a given plane P, then in its motion along the directrix it will generate the surface of a cylindroid; that is, no matter what the position of the generatrix, A_nB_n must be parallel to P.

Fig. 247 shows the projections of a cylindroid generated by a straight line AB moving along directrices KL and MN. Since in this case the plane director P is a horizontal projecting plane, the

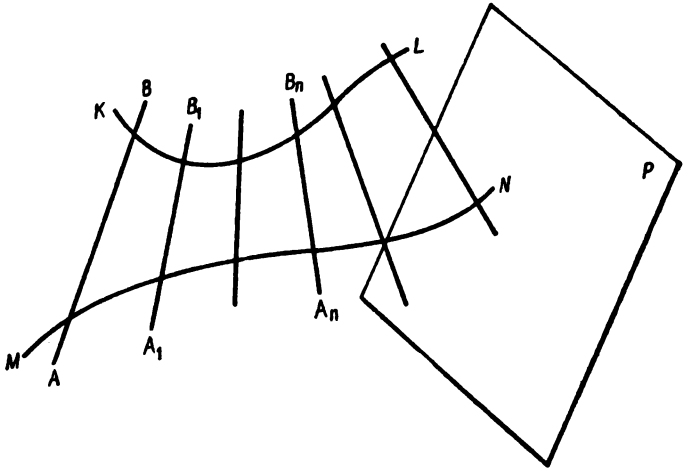


Fig. 246

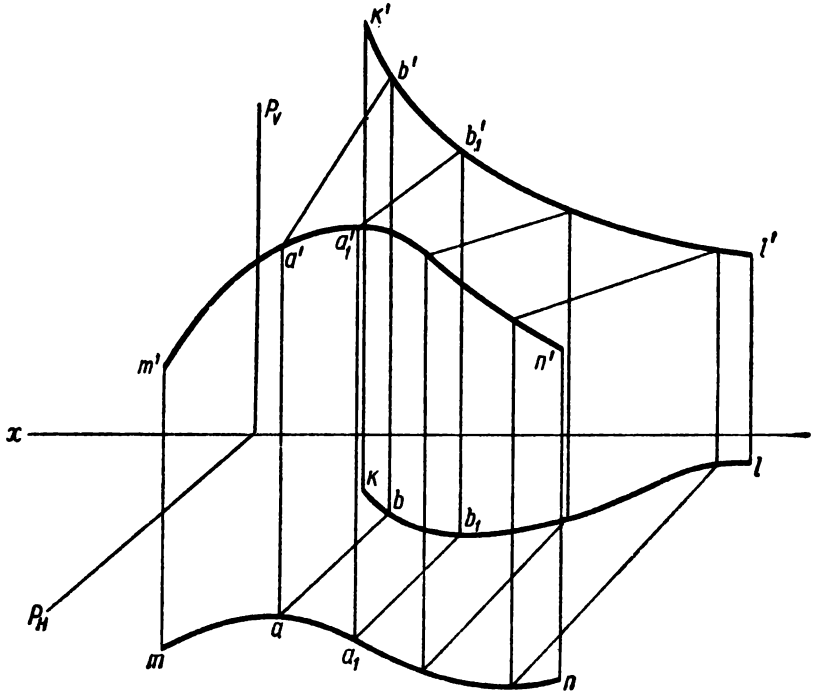


Fig. 247

horizontal projections of the generatrix in any position must be parallel to the horizontal trace of plane P_H.

The surfaces of certain vaults and arches illustrate the cylindroid. Fig. 248 shows an arch whose surface is the locus of straight lines parallel to the horizontal plane and intersecting two curved lines KLM and $K_1L_1M_1$. The projections of the straight-line generatrix of the cylindroid in two positions are labelled AB and A_1B_1.

The surface of a conoid is determined by a plane director and two directrices, one of which is straight-line and the other curved-line. The latter one may be a space curve or a plane curve, in which case the plane of the curve must not coincide with the straight-line directrix.

Remaining parallel to the director plane P, the straight-line generatrix AB (Fig. 249) will describe a conoid surface as it moves along the directrix. For any position of the generatrix, A_nB_n must be parallel to P.

Fig. 250 shows in projections the construction of one of the positions of the straight-line generatrix AB of a conoid defined by the directrices KL and MN and a plane director P. On the curved-line directrix KL an arbitrary point A is taken through which the generatrix of the conoid must pass. The position of the generatrix is determined by the fact that it must, first, be parallel to the plane P and, second, intersect the second directrix (straight line MN). The locus of all possible straight lines which pass through point A and are parallel to plane P is a plane which is parallel to P and passes through the same point A. Just such a plane P_1 is constructed orthographically by means of the horizontal line AC.

It remains to find the point of intersection of the straight-line directrix MN with the constructed plane. This is point B, which, together with A, is what determines one of the positions of the generatrix of the conoid. In similar fashion we can construct the projections of a generatrix that intersects the curve KL in other points.

When the curved-line directrix of a conoid is a cylindrical helix, the straight-line directrix is the cylinder axis, and the plane director is perpendicular to the axis of the cylinder, the conoid becomes a right helicoid (which we have already discussed theoretically and in its applications).

The surface of a hyperbolic paraboloid is determined by a plane director P and two noncoplanar straight-line directrices. The straight-line generatrix AB moving along the directrices (and remaining parallel to the plane director P) describes the surface of a hyperbolic paraboloid (Fig. 251). Hence, in this case as well, any position of the generatrix (A_nB_n) must be parallel to P.

The following are some properties of this surface.

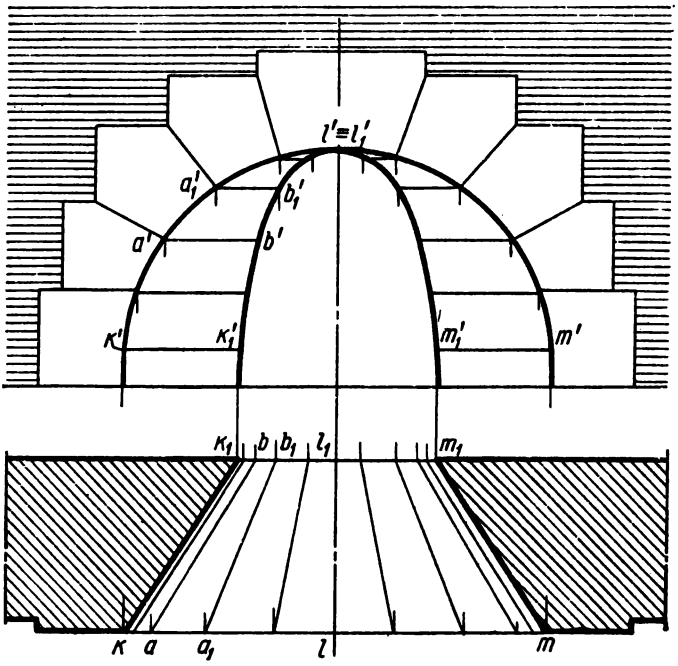


Fig. 248

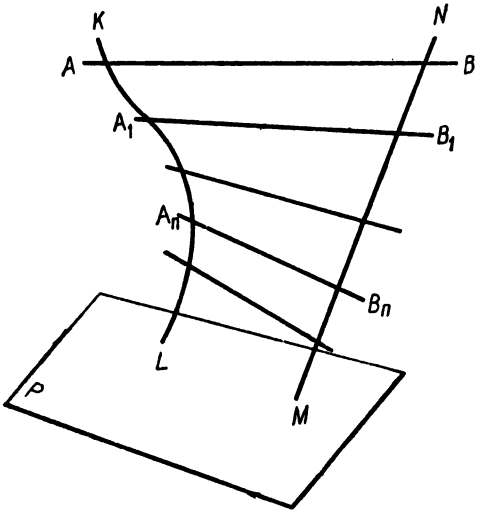


Fig. 249

1. The surface of the hyperbolic paraboloid is a double-ruled surface. Like the hyperboloid of one nappe, it also contains two families of straight-line generatrices.
2. Two generatrices of one family are skew lines.
3. Two generatrices of different families intersect.
4. Each family of generatrices has its plane director.

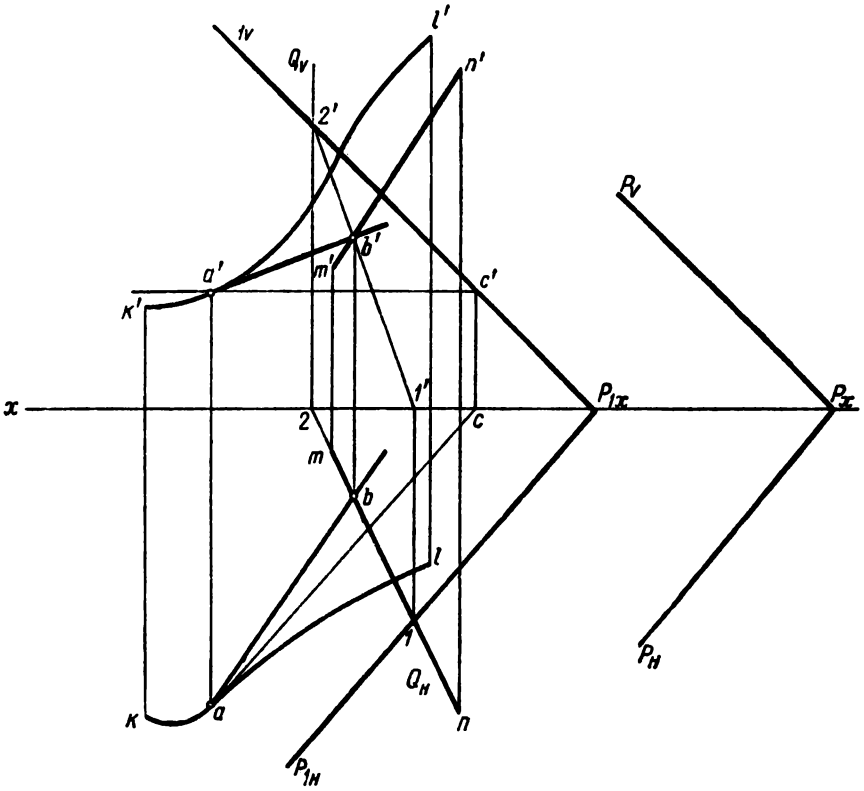


Fig. 250

If a hyperbolic paraboloid is given by two directrices MN and M_1N_1 (see Fig. 251) and a plane director P, the second family of straight-line generatrices and the second plane director are found as follows.

First construct the generatrices of the family whose plane director is plane P. To do this, take a certain plane P_1 parallel to P. Points *1* and *2* of the intersection of plane P_1 with directrices MN and M_1N_1 will determine the position of one of the generatrices, A_1B_1. By

translating plane P_1 parallel to plane P, it is possible to construct a whole series of generatrices of the first family, which are shown in Fig. 251 by the straight lines A_2B_2, A_nB_n. Besides the first family of generatrices thus created, the surface of a hyperbolic paraboloid contains two straight-line directrices MN and M_1N_1, which must belong to the second family. This latter statement is based on the fact that the hyperbolic paraboloid is only a double-ruled surface.

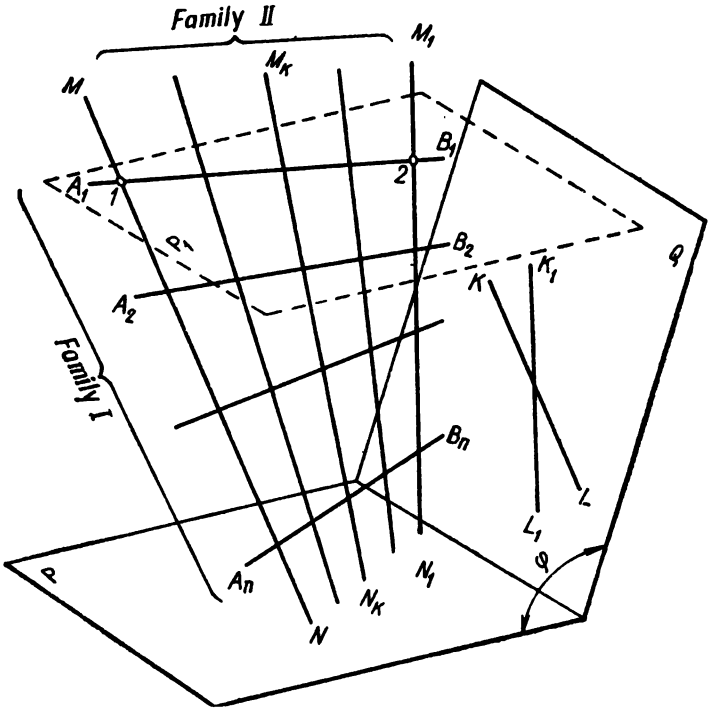


Fig. 251

The second plane director will be determined by two intersecting straight lines that are respectively parallel to MN and M_1N_1.

Fig. 251 depicts plane Q determined by the straight lines KL (parallel to MN) and K_1L_1 (parallel to M_1N_1). Plane Q is the second plane director. The second family of generatrices will be obtained by the motion of one of the directrices of the first family (say, MN) along any two generatrices of the first family (say, A_1B_1 and A_nB_n). The motion of line MN should be such that it is always parallel to plane Q; that is, M_kN_k is parallel to Q.

Depending on angle φ, which is formed by the plane directors, we distinguish a right hyperbolic paraboloid ($\varphi=90°$) and an oblique hyperbolic paraboloid ($\varphi\neq 90°$).

Besides the foregoing method of specifying the surface of a hyperbolic paraboloid by two straight-line directrices and a plane director, this surface may be determined by a warped quadrilateral.*

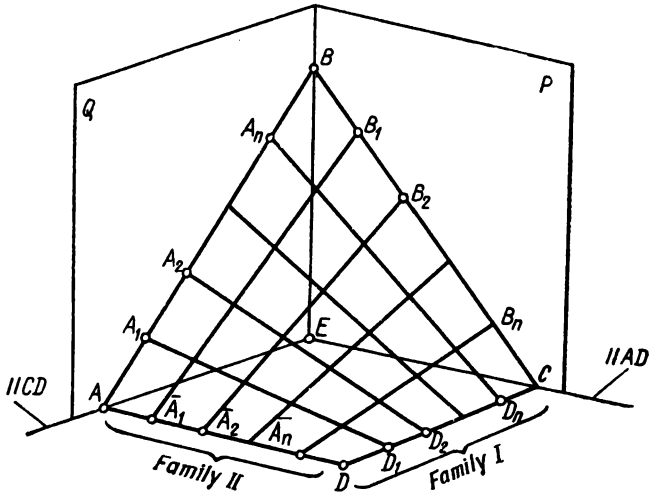


Fig. 252

Indeed, take a warped quadrilateral $ABCD$ (Fig. 252). Two opposite sides (say AB and CD) may be taken as directrices; then the two other sides, AD and BC, define the position of the plane director. In this figure it is plane P, which coincides with side BC and line CE, the latter being parallel to AD.

The first family of generatrices of the hyperbolic paraboloid may be obtained by the movement of side AD along the directrices AB and CD when the line AD remains parallel to plane P. The second family of straight-line generatrices is obtained by the motion of AB or CD along the other two opposite sides of the quadrilateral (in our case, along AD and BC) when side AB or CD travels parallel to plane Q, which, being a plane director, must be parallel both to AB and CD. In Fig. 252, plane Q coincides with side AB and passes through the straight line AE parallel to CD. The generatrices of the second family are the straight lines AB, A_1B_1, A_2B_2, ..., A_nB_n and CD.

* A warped quadrilateral is a quadrilateral having opposite sides skewed.

The orthogonal projections of a hyperbolic paraboloid given by the warped quadrilateral $ABCD$ are shown in Fig. 253. In this case, the plane directors P and Q are horizontal projecting planes, and their traces P_H and Q_H will be parallel to ad and ab, respectively.

To see why, we consider the construction of a plane director of two skew lines AB and CD, the horizontal projections of which are parallel (Fig. 254).

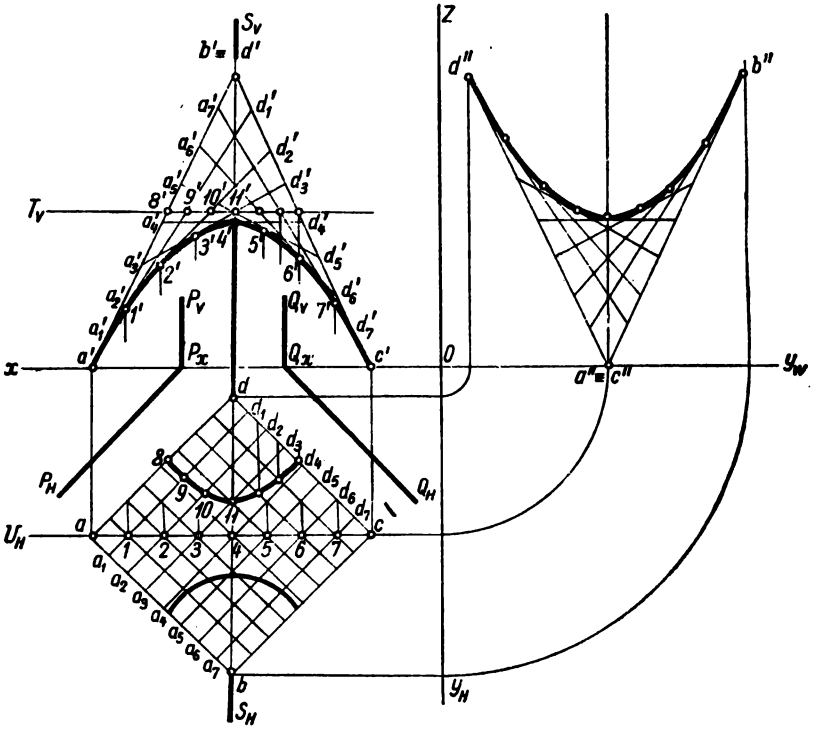


Fig. 253

The plane director Q is determined by the straight lines A_1B_1 and C_1D_1 intersecting in point K; these lines are parallel, respectively, to AB and CD. The horizontal projections of the lines A_1B_1 and C_1D_1 merge into a single straight line, which can occur only in the case of straight lines belonging to a projecting plane.

Consequently, the plane directors of the opposite sides of the given quadrilateral $ABCD$ will also be projecting planes because the horizontal projections of these sides are parallel. In Fig. 253 we have

constructed two families of straight-line generatrices of a hyperbolic paraboloid. The generatrices of the first family are designated A_1D_1, A_2D_2, A_3D_3, ... These are straight lines parallel to plane P and intersecting sides AB and CD. The latter are the directrices for the first family of straight lines. The projections of the second family of generatrices that are parallel to plane Q are not shown in the orthographic representation.

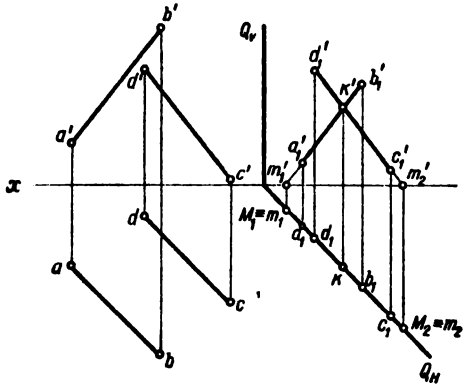


Fig. 254

The curved-line contours of the vertical and profile projections are parabolas, which in this case are the envelopes of projections of the straight-line generatrices. At the same time, we may regard each parabola as a section of the surface by the U and S planes, respectively.

Then parabola *A-I-II-III-...* may be constructed as the locus of points of intersection of the straight-line generatrices AD, A_1D_1, A_2D_2, ... with plane U parallel to V.

If the hyperbolic paraboloid is intersected by a horizontal plane T, then we get a hyperbola. Points *VIII*, *IX*, *X* and *XI* of one of the branches of the hyperbola are found as the points of intersection of lines AD, A_1D_1, A_2D_2, and A_3D_3 with plane T.

And so this surface may have straight lines and also parabolas and hyperbolas, which accounts for the name: hyperbolic paraboloid.

We shall now illustrate the applications of hyperbolic paraboloid surfaces in building construction.

The surface of a hyperbolic paraboloid is encountered in the embankments of what are called open cuts. According to specifications they are made two metres deep in treeless areas (where snowbanks pile up) with a width no less than 10 metres from the axis of the adjacent lane on either side of the roadway unless the project calls

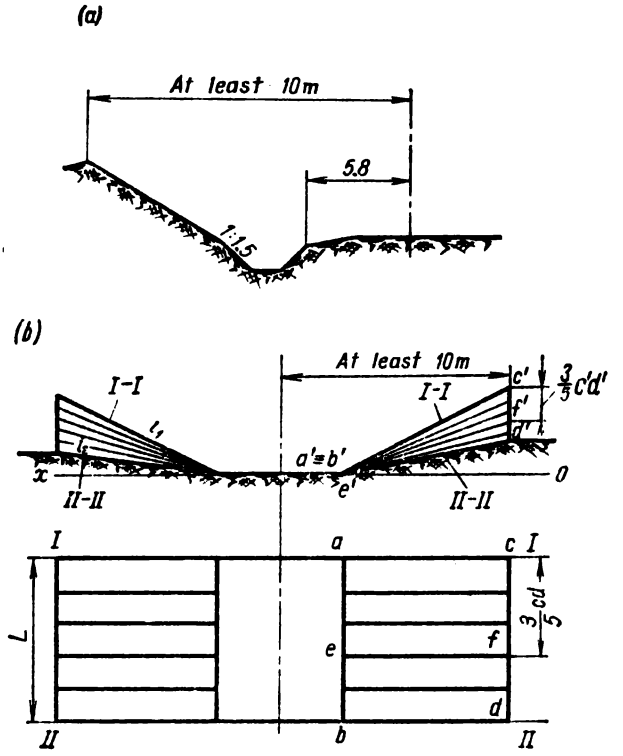


Fig. 255

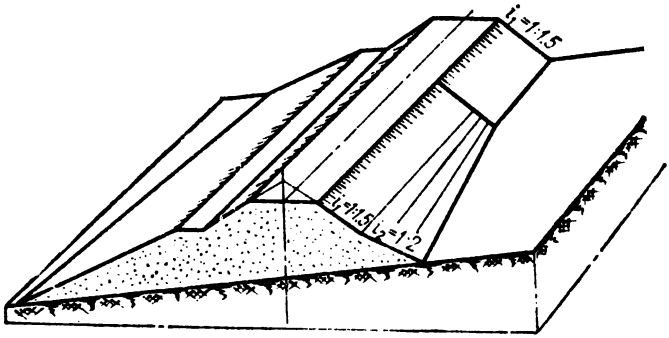


Fig. 256

for hedges along such cuts. These embankments (see Fig. 255*a*) are useful in that they utilize wind energy for fighting snowdrifts.

Fig. 255*b* depicts simplified cross-sectional profiles and an open cut in plan on a section of length L (to simplify the drawing, drain ditches are not shown).

The slope of the embankments of the cut, when passing from section *I-I* to section *II-II*, changes from i_1 to i_2; in our case, $i_1 > i_2$.

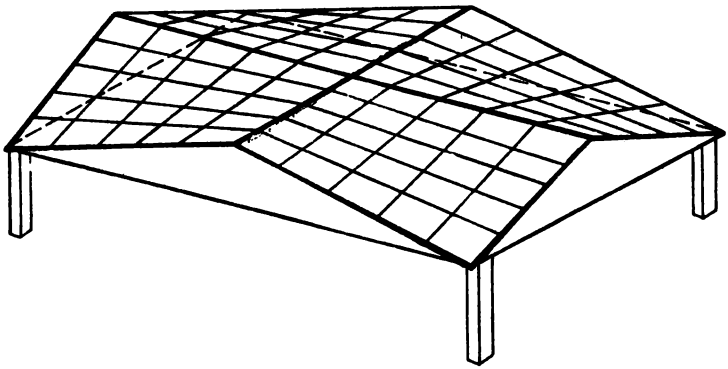


Fig. 257

In this example the V plane, which is perpendicular to the axis of the roadbed, serves as a plane director. For the right hand slope of the cut, the road-edge AB and the line of intersection of the slope with the earth's surface (we have in view a sloping plane surface—this is a straight line CD) serve as directrices.

Since the generatrices of a hyperbolic paraboloid must be parallel to the plane director, the horizontal projections of each of them will be parallel to the x-axis.

Transition sections of the roadbed of a railway line are also hyperbolic paraboloid surfaces. Such sections may be utilized in designing high flood-plain embankments where the slope towards the water should not be steeper than 1 : 2 (Fig. 256).

The hyperbolic paraboloid is used for roofing without intermediate supporting structures. The simplest is a covering bounded by a single warped quadrilateral on a square plan. Fig. 257 shows four such coverings joined together and resting on supports at each corner.

CHAPTER VIII

INTERSECTION OF CURVED SURFACES WITH A PLANE AND A STRAIGHT LINE

Sec. 45. A Curved Surface Cut by a Plane

The section of a curved surface by a plane may be regarded as the locus of the points of intersection of lines drawn in the surface with the given plane. These may be any kind of lines However, to ensure accuracy and simplify construction it is best to choose the simpler types of line. For ruled surfaces these are straight-line generatrices; for surfaces of revolution, circles (parallels of a surface of revolution)

Fig. 258 illustrates the construction of a section of the surface of a hyperbolic paraboloid by a horizontal projecting plane Q. The hyperbolic paraboloid is generated by the motion of straight line AB, parallel to the V plane, along the skew lines AD and BC. Points I, II, III, ... are the points of intersection of the straight-line generatrices of the plane with plane Q. Their locus defines the desired sectional curve. Similar cases were examined in Figs 236 and 240.

The construction of a line along which a plane intersects a cylindrical and a conic surface also reduces to solving the same kind of position problem on the intersection of a straight line with a plane. As a rule, it is best to start these constructions by determining the so-called reference points of the required line and then add intermediate points. The reference points include the highest and lowest points of the section and points on the contour curves of the projections. The first two will suggest at what altitude on the surface we are to locate the intermediate points, while the points on the contour curve of each projection separate the visible part of the sectional curve from the hidden part.

Fig. 259 illustrates the construction of the uppermost and lowest points of a section of an oblique cylinder by plane P. Transforming plane P into a projecting plane and constructing a new vertical projection of the cylinder on plane V_1, we note that the maximal magnitude of the z-coordinate has a point I, and the minimal value will be at point II Their vertical projections in the V_1/H system are defined by the intersection of P_V with the contour line of the cylinder projection on plane V_1. A change-over to the original V/H system required the construction of generatrices that contained the established points. The horizontal trace of one of them is denoted by A, the other by B. The following four reference points (III, IV,

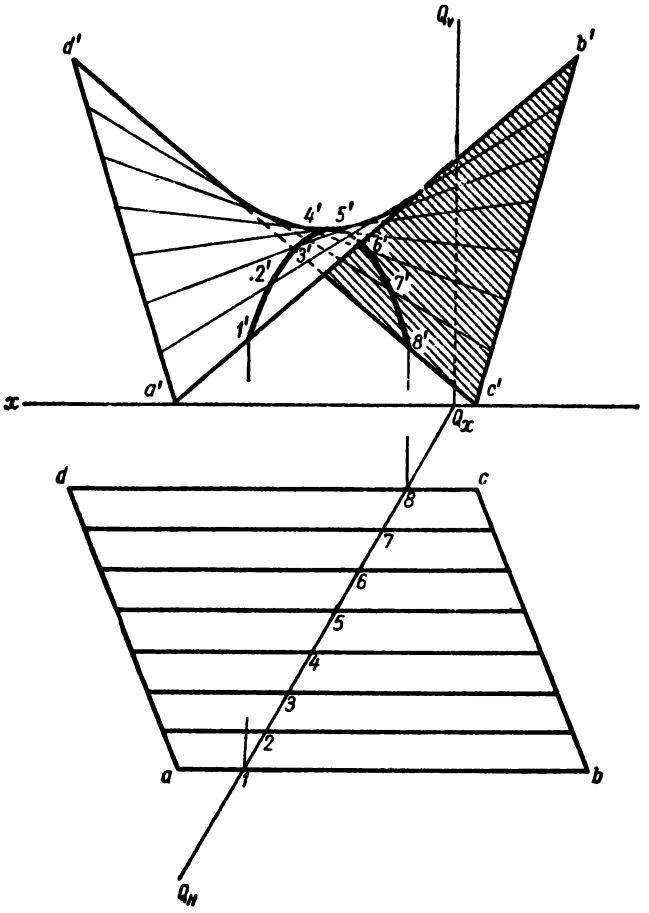


Fig. 258

V, *VI*) are points of intersection of the contour generatrices of the cylinder with plane *P* (Fig. 260). To determine these points, through the contour generatrices pass horizontal projecting planes *Q*, Q_1, Q_2, Q_3, which (all parallel to each other) intersect the given plane *P* along parallel lines. Similarly we find the remaining points of the required section.

The problem of constructing lines of intersection of a ruled surface by a plane is greatly simplified if the given plane is a projecting plane.

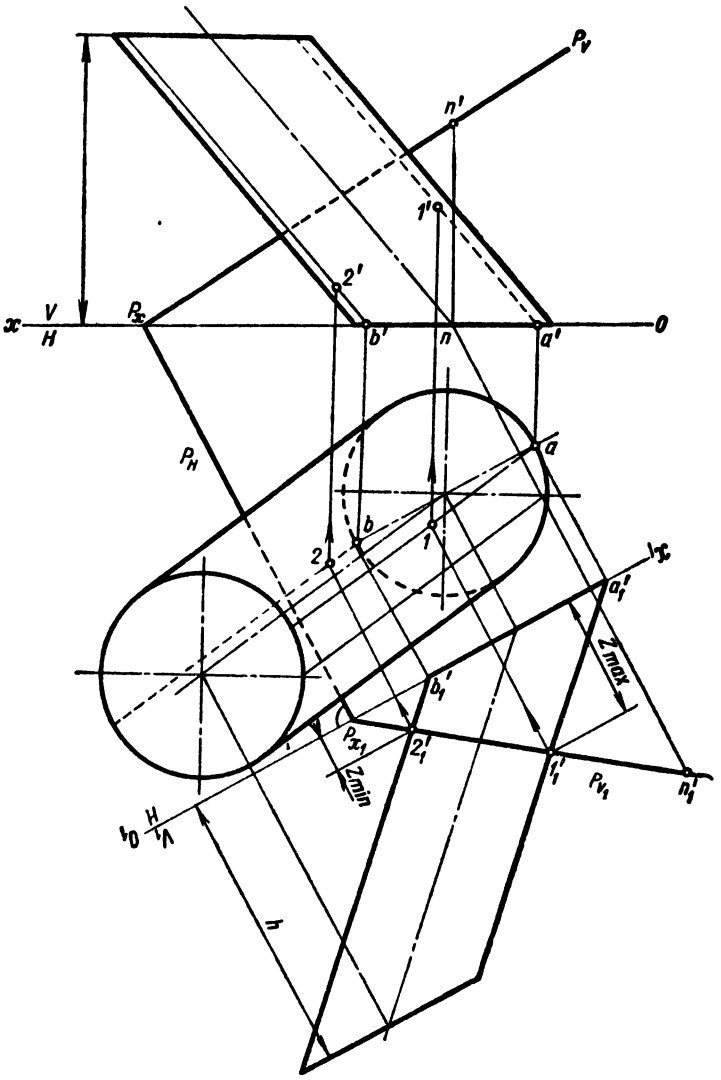

P_V
n'
$1'$
$2'$
x
V
H
P_x
b'
n
a'
0
P_H
a
b
2
1
l_x
a_1'
z_{max}
b_1'
P_{x_1}
$2_1'$
z_{min}
$1_1'$
P_{V_1}
n_1'
l_0
H
h

Fig. 259

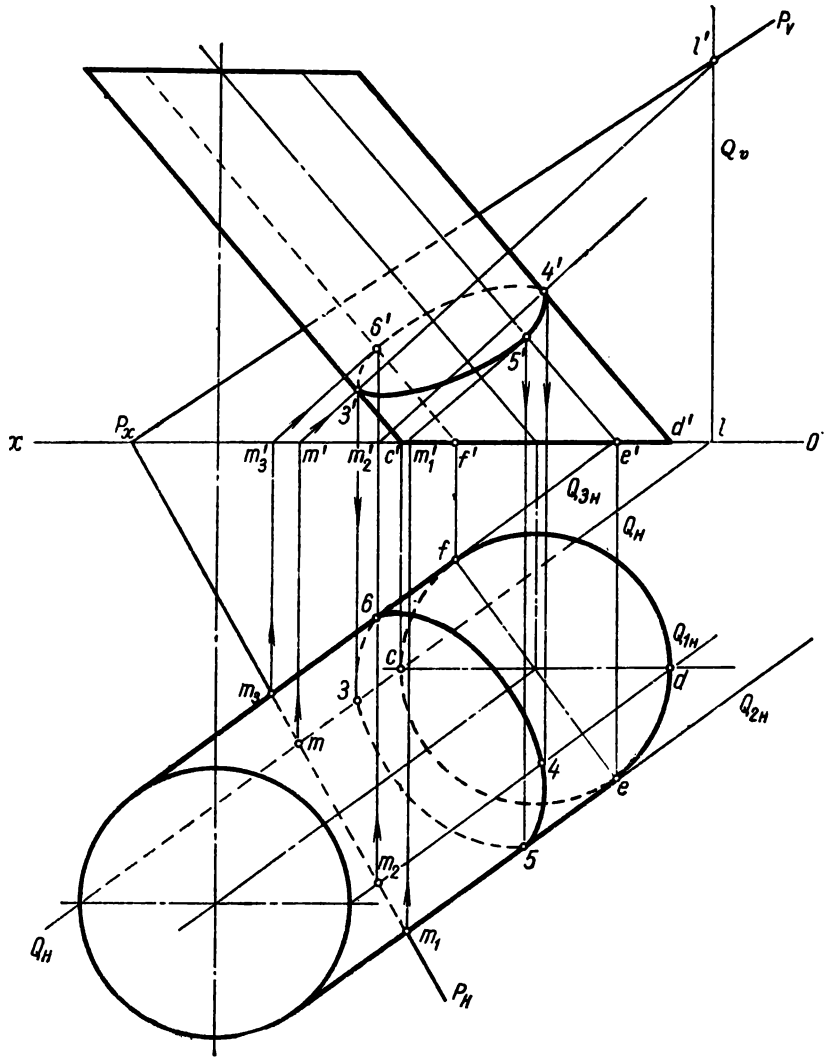

P_V
l'
Q_v
$4'$
$6'$
$5'$
$3'$
x
P_x
m_3'
m'
m_2'
c'
m_1'
f'
d'
l
0
e'
Q_{3H}
Q_H
f
6
Q_{1H}
c
d
m_3
3
m
Q_{2H}
4
e
m_2
5
m_1
Q_H
P_H

Fig. 260

Fig. 261 illustrates the construction of the projections of a cut in a wooden support. Planes P and Q are at an angle to the axis of the cylindrical support and intersect it in elliptical segments, the vertical projections of which coincide with like traces of the projecting planes, while the horizontal projections are circular segments. In the construction of profile projections of the segments, the distance

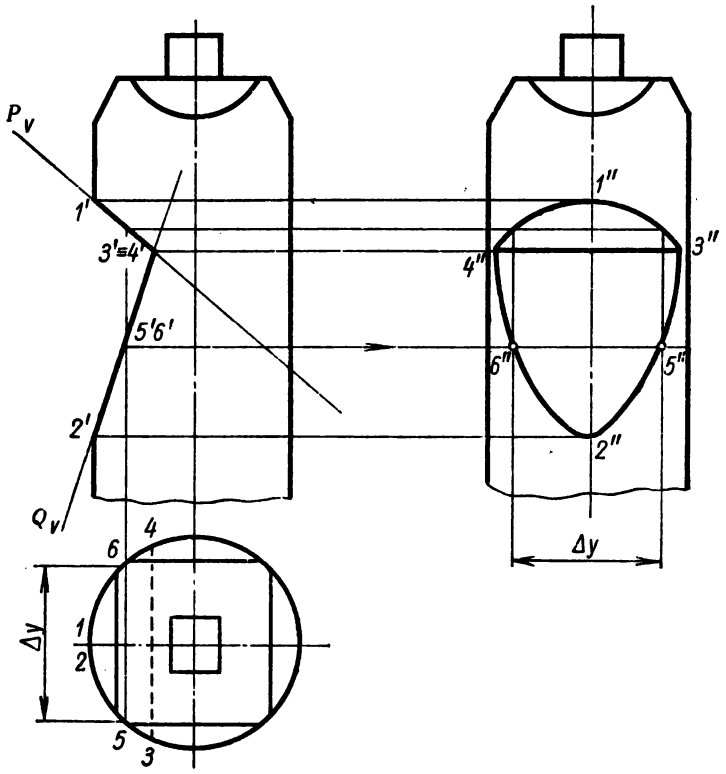


Fig. 261

between two symmetrical points of the elliptic arcs was determined in a transversal direction (that is, perpendicular to plane V) by the chord Δy, the length of which was measured on the horizontal projection.

The next example is devoted to the construction of the line along which a surface of revolution is cut by an oblique plane P (Fig. 262).

We shall first show that the desired curve and its projection on a plane perpendicular to the axis II_1 (in our case, on plane H) are lines

that have an axis of symmetry To do this we shall have to deviate somewhat from the above sequence for locating points of intersection. First, instead of locating the uppermost and lowest points, we establish two intermediate points lying on a single parallel of

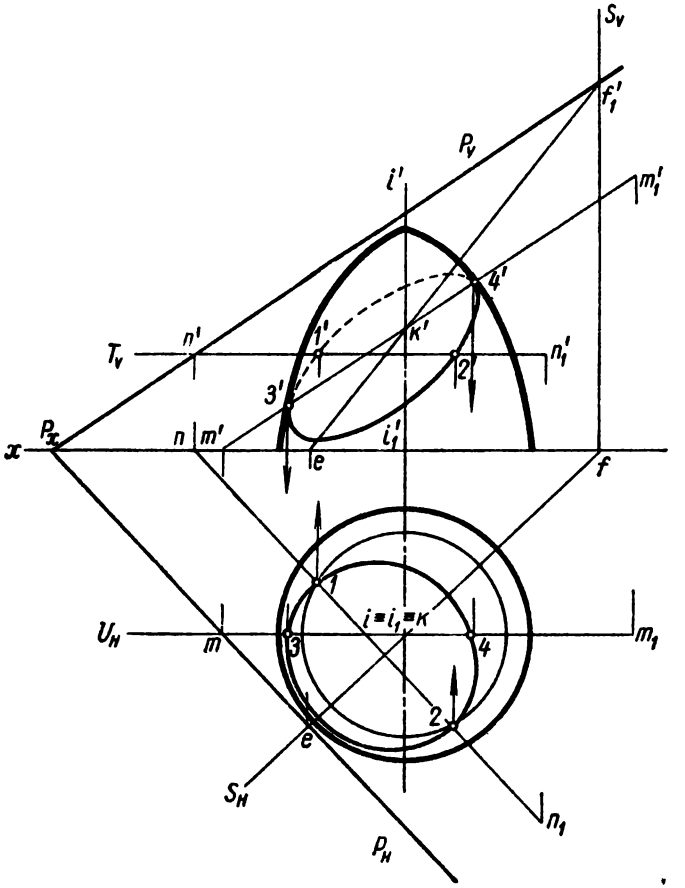


Fig. 262

the surface of revolution. Fig. 262 shows a parallel obtained by intersecting the surface with plane T, which is perpendicular to axis II_1. The auxiliary plane T will cut the given plane P along the horizontal line NN_1. Since both the parallel of the surface of revolution and the horizontal line NN_1 lie in the same plane T, they intersect in points I and II which will belong to the desired line. The points obtained are symmetrical to one another about plane S,

which is perpendicular to the chord *I-II* and passes through its midpoint. Note that since plane *S* is the locus of points equidistant from the extremities of chord *I-II*, it will pass through the axis of the surface of revolution, all the points of which are also equidistant from points *I* and *II*.

It is obvious that for any other pair of points lying at the extremities of the chords of other circles but parallel to chord *I-II*, plane *S* will likewise be a plane of symmetry. Hence, the sectional curve of the surface of revolution cut by plane *P* is a symmetrical curve, whose axis of symmetry is the intersection line of planes *S* and *P*: the straight line *EF*. This axis of symmetry is the steepest line in plane *P*; the latter line intersects the axis of the surface of revolution.

The projection of the sectional curve onto a plane perpendicular to the axis II_1 will also have the property of symmetry, since the line-segments connecting the intersection points that are symmetrical about one another are parallel to the same plane.

In this case the highest and lowest points must be symmetrical to themselves and, consequently, each of them must be located on the axis of symmetry of the sectional curve. This conclusion suggests a simple way of defining the points·

(1) draw the steepest line in plane *P*, *EF*, which intersects the axis of the surface of revolution (Fig. 263);

(2) turn this straight line and the meridian of the surface (the plane of which coincides with *EF*) about axis II_1 to position E_1F_1; then line *EF* and the plane of the meridian will be parallel to *V*. The point of intersection (*K*) of line *EF* with axis II_1 will remain fixed, while the rotating meridian will ultimately coincide with the principal meridian, the contour line of the vertical projection of the surface of revolution;

(3) note the points of intersection of the vertical projections of the principal meridian and the rotated straight line. In Fig. 263 these are points $5'_1$ and $6'_1$;

(4) counterrevolve E_1F_1 together with the points found to the original position *EF*.

Another pair of reference points on the intersection line will be formed by points located on the principal meridian of the surface (on the contour of the vertical projection). Their construction is shown in Fig. 262, where plane *U* parallel to *V* is passed through the principal meridian. Plane *U* intersects the given plane *P* along the vertical line MM_1. In turn, the latter, being in the same plane as the principal meridian, intersects it in the required points *III* and *IV*. The vertical projections *3'* and *4'* of the points thus found separate the visible portion of the projection of the sectional curve from the hidden portion.

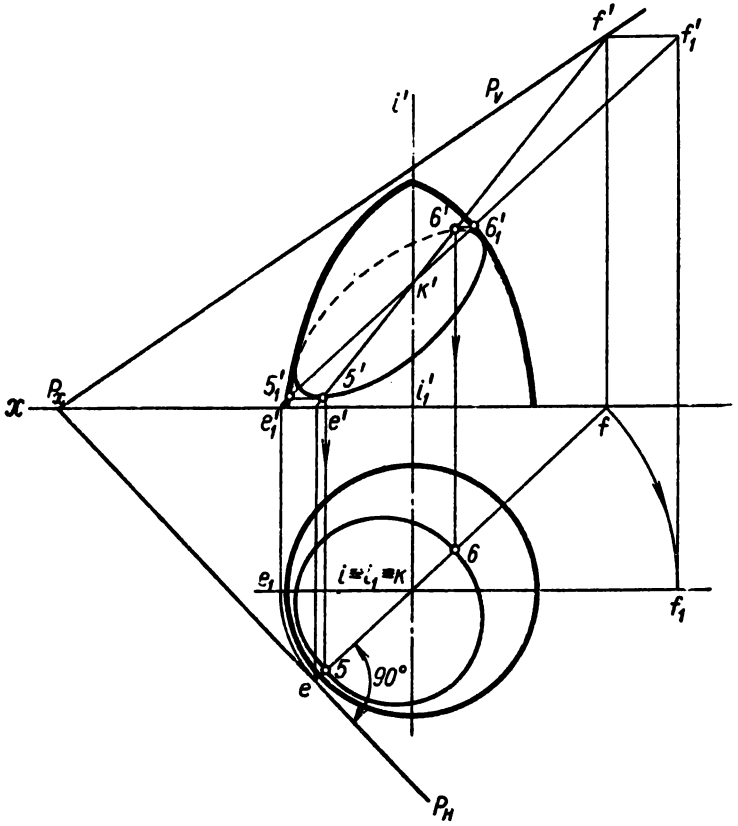


Fig. 263

Now we consider constructing intersection lines of a sphere by a projecting plane and an oblique plane.

1. Given a sphere and a vertical projecting plane (Fig. 264).

Plane Q intersects the surface of the sphere in a circle which is projected on the H plane in the form of an ellipse. In this case, the highest point (1, $1'$) and the lowest point (2, $2'$) are defined by the intersection of the vertical trace Q_V with the contour of the vertical projection of the sphere.

Points 3 and 4 that separate the visible portion of the horizontal projection from the hidden part are obtained by means of the auxiliary plane T_3 passed through the centre of the sphere parallel to the H plane. The horizontal projection of a section of a sphere by

plane T_3 (a circle) intersects the horizontal projection *mn* of the horizontal line* to yield points *3* and *4*.

Using auxiliary planes T_1 and T_2 we get other points contained in the desired section. It will be noted that plane T_2, which passes through the midpoint of line-segment *1'-2'*, enables us to locate the

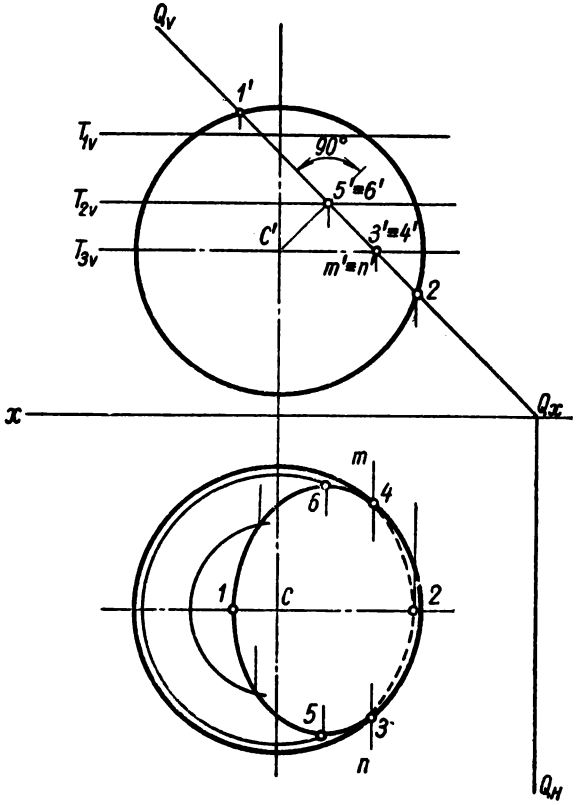


Fig. 264

major axis of the ellipse whose length must be equal to the true length of the diameter of the circle of the section; thus line-segment *5-6* is equal to line-segment *1'-2'*.

2. Given a sphere with centre Ω and an oblique plane *P* defined by two intersecting straight lines *AB* and *BC* (Fig. 265).

* The horizontal line *MN* is the line of intersection of planes T_3 and *Q*.

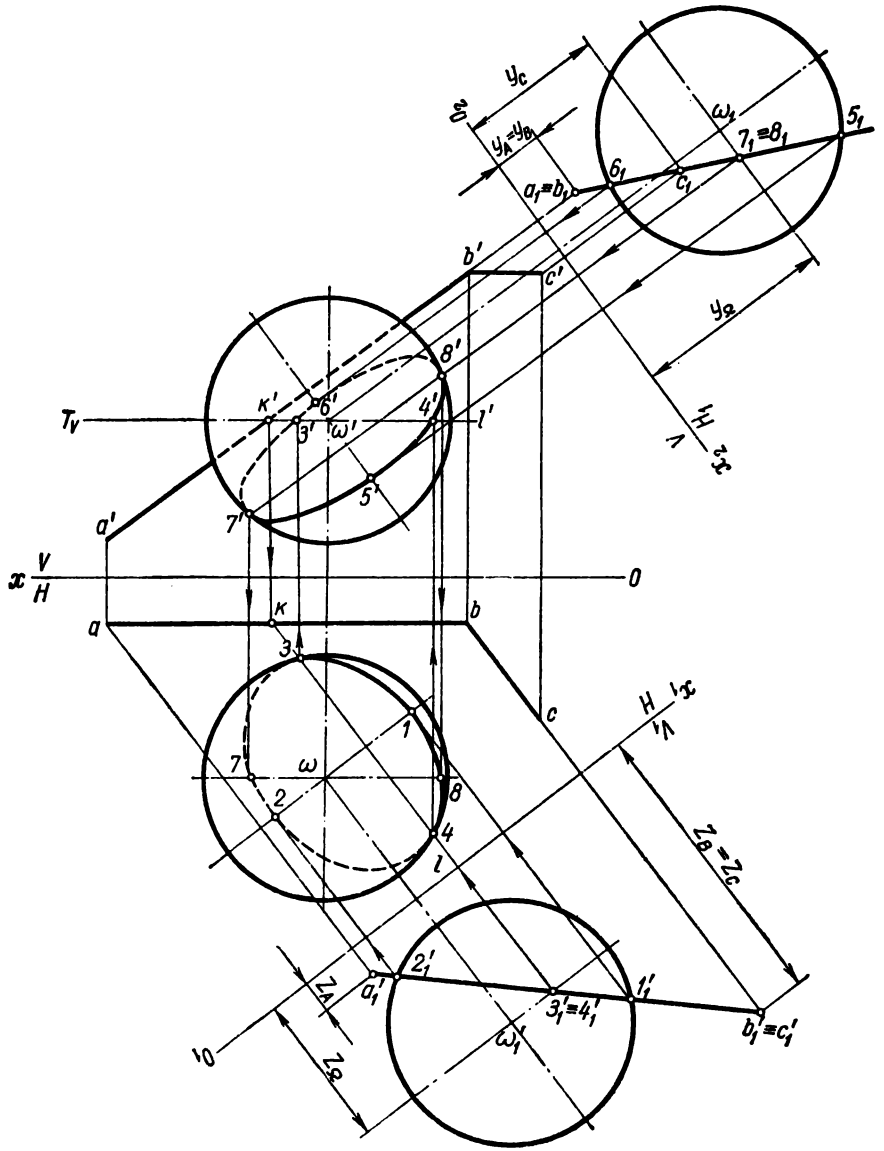

y_C
$y_A = y_B$
$a_1 \equiv b_1$
ω_1
5_1
$7_1 \equiv 8_1$
6_1
c_1
b'
c'
y_B
$8'$
T_V
k'
$6'$
$4'$
$3'$
ω'
l'
$5'$
$7'$
a'
x V H
O
k
a
b
3
1
c
7
ω
8
2
4
l
H l_x V_1
$Z_B \equiv Z_C$
$2_1'$
a_1'
Z_A
$1_1'$
$3_1' \equiv 4_1'$
$b_1' \equiv c_1'$
ω_1
l_0
Z_B

Fig. 265

In our example, AB is a vertical line (V principal line) and BC is a horizontal line of plane P.

The case at hand may be reduced to the preceding one by replacing the V plane with the V_1 plane perpendicular to BC. In the new V_1/H system the given plane is a projecting plane. The horizontal projection of the section may be constructed as shown in Fig. 264.

The highest and lowest points of the section are designated by *I* (*1*, *1'*) and *II* (*2*, *2'*), respectively. The points that are located on the contour of the horizontal projection of the sphere and separate the visible part of the horizontal projection from the hidden part are denoted by *III* (*3*, *3'*) and *IV* (*4*, *4'*).

It will be noted that these points (*III* and *IV*) may be located directly in the V/H system by the aid of plane T that passes parallel to H through the centre of the sphere.

Plane T intersects plane P along the horizontal line KL (kl is parallel to bc) and intersects the sphere along the equator.

The intersection of the horizontal projections of the equator and the horizontal line define the points of visibility: *III* and *IV*.

The vertical projection of the section may be obtained independently of the already constructed projection of the section on plane H. To do this, pass from the V/H system to the V/H_1 system. The new H_1 is perpendicular to the vertical line AB of the given plane.

Subsequent constructions do not essentially differ from the preceding ones.

Note that V and *VI* indicate points which are respectively farthest from and nearest to the V plane. Points *VII* and *VIII* are located on the principal meridian of the sphere and define the boundaries of visibility of the vertical projection of the intersection line.

In large-size drawings these points are sometimes not sufficient for an accurate construction of the projections of the sections. Intermediate points may be determined by means of additional parallels of the sphere, as indicated in the case of an oblique surface of revolution (see Fig. 262, points *I* and *II*).

We conclude with the so-called conic sections that find extensive applications in engineering. Fig. 266 depicts three sections of a right circular cone* by a variety of planes. Dispensing with the proofs, we make the following statements:

(1) intersecting a right circular cone with a plane that is not parallel to any of its generatrices yields an elliptical section (Fig. 266*a*);

(2) intersecting the same cone with a plane parallel to one of its generatrices yields a parabola (Fig. 266*b*);

* A cone, the base of which is a circle, is a right cone if its altitude passes through the centre of the base; otherwise it is an oblique cone.

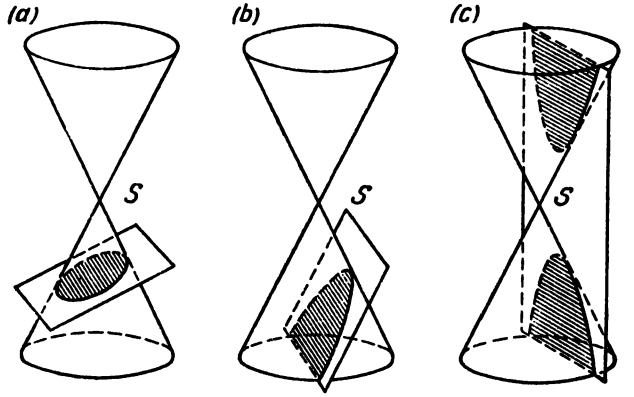


Fig. 266

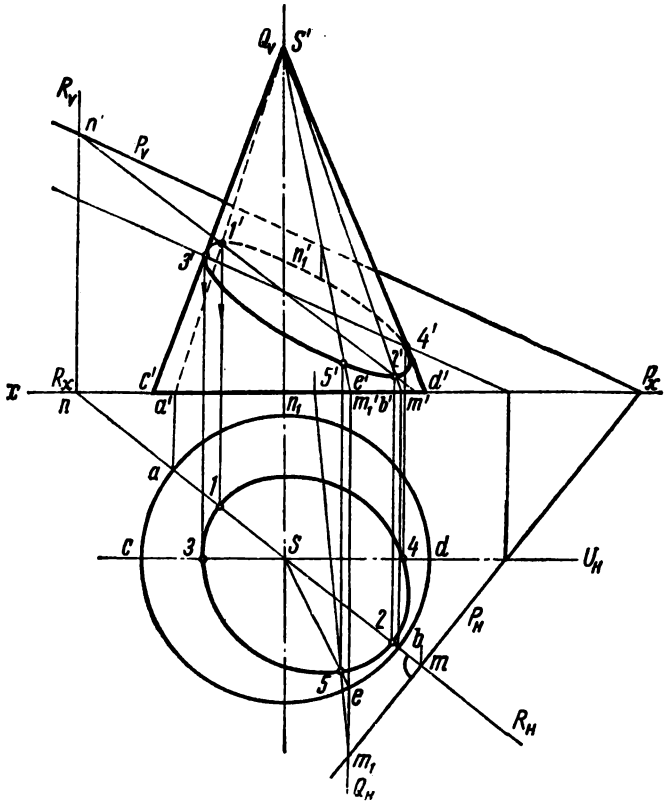


Fig 267

(3) if the cutting plane is parallel to two generatrices of the cone (a special case is when it is parallel to the axis), the section obtained is that of a hyperbola (Fig. 266*c*).

We also note that a cutting plane passing through the vertex intersects the cone along the generatrices.

The foregoing statements hold for oblique, circular, and elliptical cones as well; that is, for a cone whose surface in Cartesian coordinates is expressed by a second-degree equation.

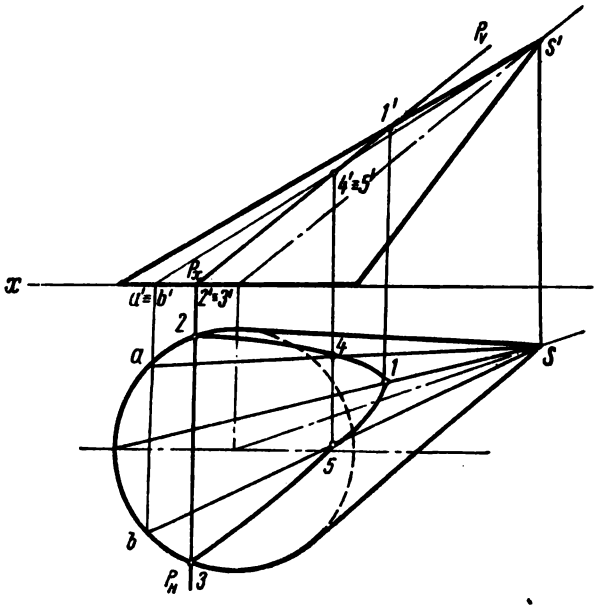


Fig. 268

Some of the practical applications of surfaces whose generatrices are conic sections were described in Secs. 40 and 41. Figs. 267 and 268 illustrate the construction of orthogonal projections of conic sections.

In constructing the section of a right circular cone by an oblique plane P, first find the highest and lowest points (*I* and *II*). For this purpose, through the cone axis pass plane R perpendicular to trace P_H. Line MN along which plane R intersects P will be the axis of symmetry of the desired section. The self-symmetrical highest and lowest points of the sectional figure are determined by intersection of the straight line MN with the generatrices SA and SB, along which plane R intersects the cone. Then, by means of the auxiliary plane U parallel to V we construct points *III* and *IV* located on the contour

generatrices SC and SD. We then determine the intermediate points at which the straight-line generatrices of the cone are intersected by plane P. To illustrate (see Fig. 267), point V is constructed so as to belong to the generatrix SE.

The oblique cone shown in Fig. 268 is intersected by a vertical projecting plane P parallel to its axis. The conic section here is a hyperbola, the projections of one of the branches of which are constructed in Fig. 268. The resulting line is the locus of the intersection of the straight-line generatrices of the cone with the given plane.

Sec. 46. Intersection of a Straight Line with a Curved Surface

The problem of determining the points of intersection of a straight line with a curved surface is solved in the same sequence as was suggested for determining points of intersection of a straight line with a plane or the surface of a polyhedron. The sequence of the solution is:

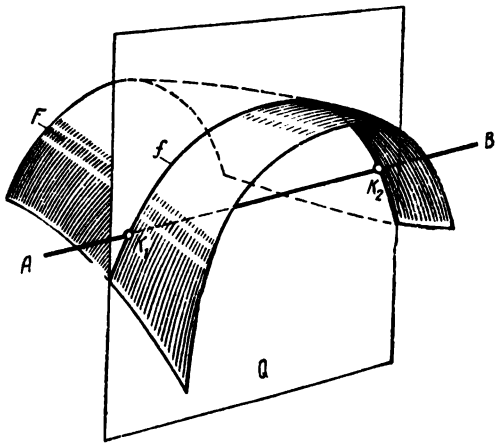


Fig. 269

(1) through a given straight line AB draw an auxiliary plane Q (Fig. 269);

(2) find the sectional line of the given surface F by the auxiliary plane Q (curve f), and

(3) determine points K_1 and K_2 (the points of intersection of the constructed section with line AB).

The number of required points depends on the type of surface and the location of the straight line relative to the surface.

To simplify construction, the auxiliary plane is chosen for each specific problem so that its line of intersection with the given surface is of the simplest type possible. For example, in locating the points of intersection of a straight line with the surface of a cone and cylinder it is convenient to make use of the most elementary cutting planes.

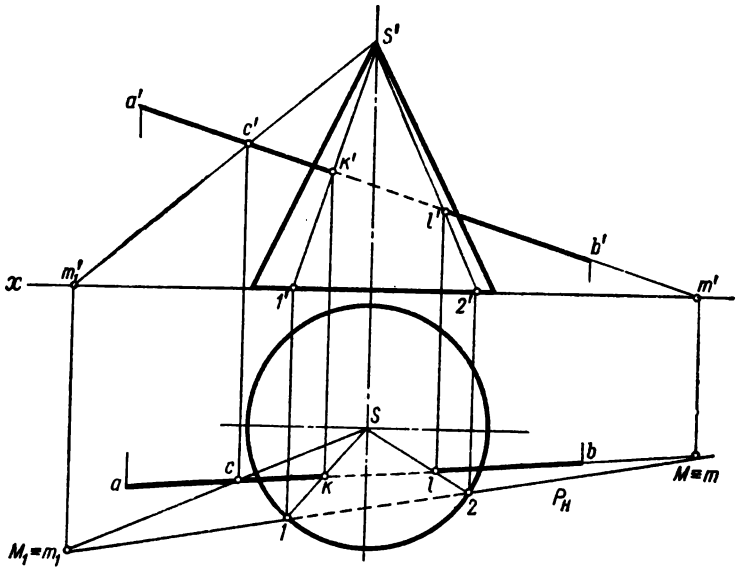


Fig. 270

As in the case of the pyramid, let an elementary cutting plane for a cone be a plane that passes through the cone vertex intersecting the cone along its generatrices.

In Fig. 270 such a plane P is defined by the given line AB and the cone vertex S. When constructing the horizontal trace P_H, it was necessary to pass from specifying the plane by a point and a straight line to specification by two intersecting straight lines AB and SC (C is an arbitrary point on line AB). Their horizontal traces M and M_1 define P_H. Joining points I and II, in which P_H cuts the cone base, with vertex S, we get a conic section: triangle $ISII$. Constructing the projections of this triangle, we note the points of intersection of the projections of its sides with the projections of the given straight line AB. The resulting points K and L will be the required ones.

With an elementary cutting plane we find the points of entry and exit of AB on the given surface of the cylinder (Fig. 271).

As for a prism, the elementary cutting plane for a cylinder must intersect the cylinder in a parallelogram, and its lateral surface,

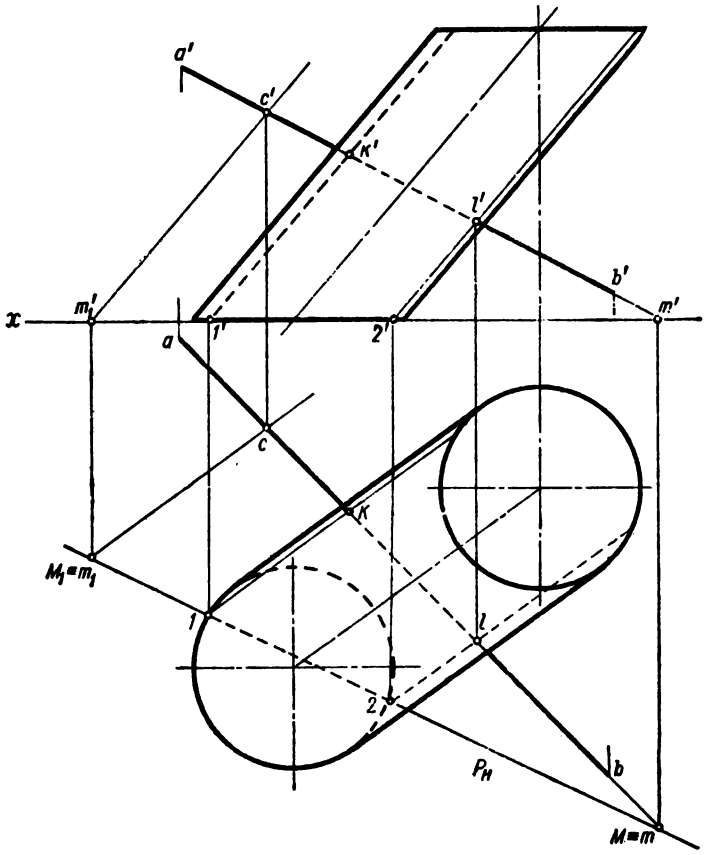


Fig. 271

along generatrices. In Fig. 271 the elementary cutting plane P is defined by two intersecting straight lines: the given one, AB, and an auxiliary line, CM_1, which is drawn parallel to the generatrices of the cylinder. The horizontal trace of this plane is defined by two points, M and M_1, which are the traces of the straight lines AB and CM_1 Draw generatrices through points I and II, in which P_H intersects the cylinder base. Points K and L, which lie on the given

straight line AB and the established generatrices of the cylinder, are the required ones.

Fig. 272 illustrates a determination of intersection points K_1 and K_2 of line AB and a surface of revolution.

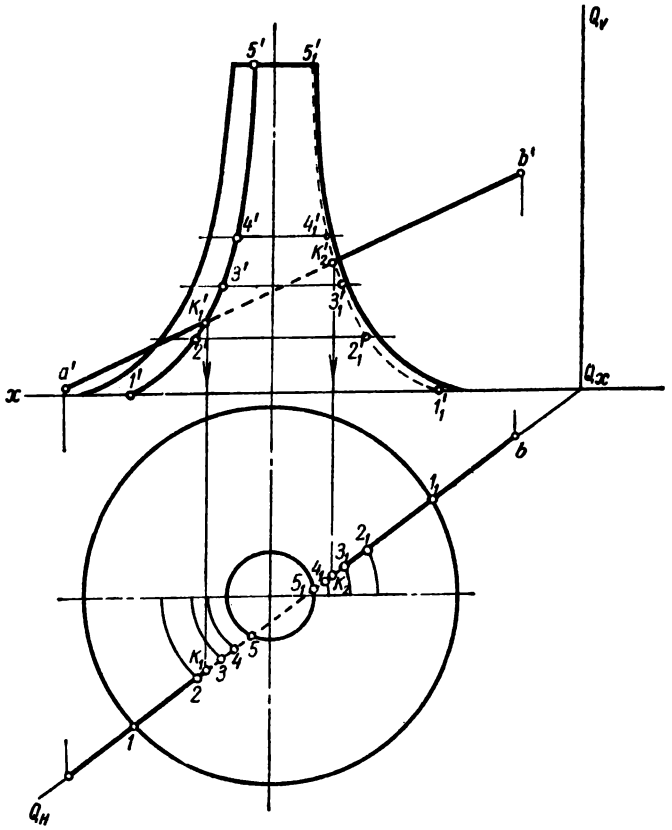


Fig. 272

To construct the projections of the desired points, draw the horizontal projecting plane Q through the straight line and determine two lines (I-II-III-IV-V and I_1-II_1-III_1-IV_1-V_1) of intersection of this plane with the surface of revolution. The curves are constructed with the aid of parallels, as described in Sec. 45.

Fig. 273 shows the construction of K and L, in which line AB intersects the surface of a sphere. In this case, through AB we pass an auxiliary plane Q which intersects the sphere in a circle. In

order to dispense with the construction of the projection of the circle on the V plane, the problem is solved by the method of replacing projection planes.

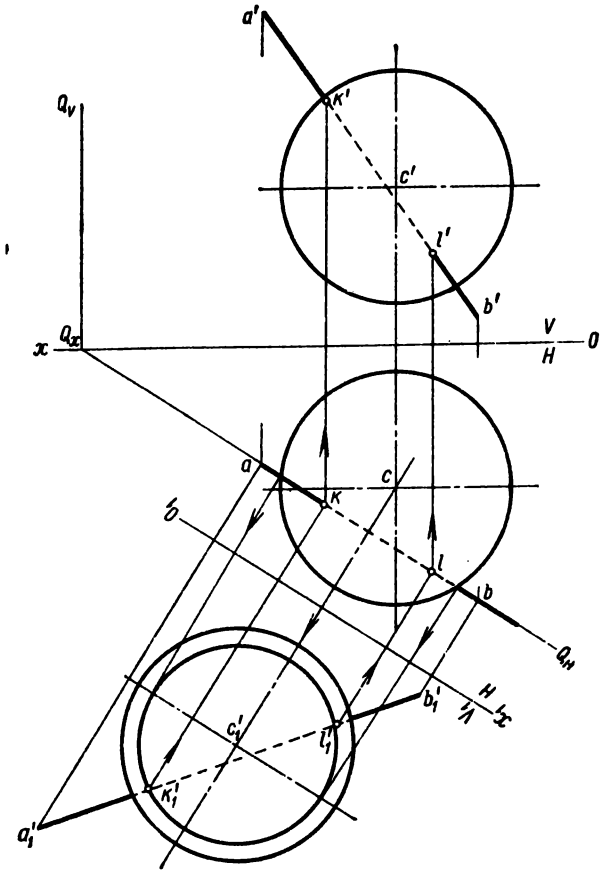


Fig. 273

The new plane V_1 is perpendicular to H and parallel to AB.

As a result, plane V_1 is parallel to the auxiliary plane Q and the intersection line of plane Q and the sphere is projected full size on V_1. Construction of the projection of the intersection line on V_1 requires less time and yields a more accurate result than the construction of the projection of that line on the V plane.

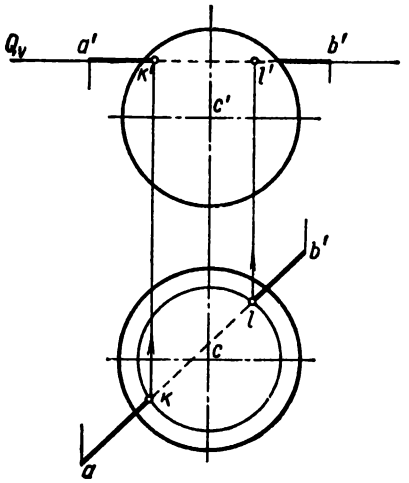


Fig. 274

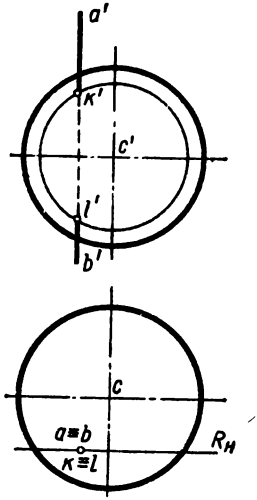


Fig. 275

It is well to bear in mind that the suggested replacement of the projection planes is advisable only when line AB is an oblique straight line.

However, in special cases where the straight line is parallel to H or V (Figs. 274 and 275) it is not advisable to change the planes.

CHAPTER IX

PLANES TANGENT TO CURVED SURFACES

Sec. 47. Fundamentals

Differential geometry provides proof that a plane tangent to a surface F at a point A is the locus of straight lines tangent to any curves passing along the surface through the given point.

Consequently, the position of a plane P tangent to a surface at a given point A may be defined by two straight lines AB and AC, each of which is tangent to a curve drawn in the plane through point A. Fig. 276 shows straight lines AB and AC tangent to curves f_1 and f_2.

The examples which will be given below will show that a plane may be tangent to a surface either at a point or along a line (straight or curved). When the plane touches a surface at a given point, it may cut the surface along one or two lines. On a surface there may be points at which it is impossible to pass a tangent plane. These points are called singular. They include the nodes of a surface, the points of the cuspidal edge, and the cusped vertices of surfaces of revolution (when a generatrix does not cut the axis of revolution at a right angle).

At singular points, a tangent plane either cannot be defined uniquely or does not exist at all. Points at which a unique tangent plane may be drawn are called ordinary points. Finally, we introduce yet another concept, the normal to a surface. This is a straight line perpendicular to the tangent plane and passing through the point of tangency. It is obvious that problems on construction of normals to curved surfaces reduce, essentially, to problems on construction of tangent planes.

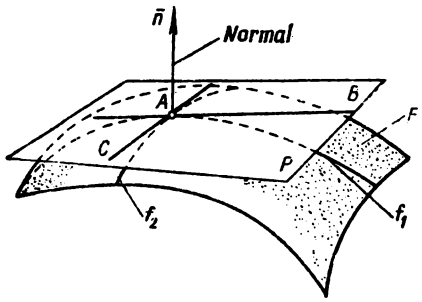


Fig. 276

Sec. 48. Planes Tangent to Ruled Surfaces

Ruled surfaces are the loci of straight lines (generatrices). A plane tangent to a ruled

surface at an arbitrary point on a given generatrix will pass through the generatrix. This is due to the fact that each generatrix is its own tangent.

In the general case, a normal (to a surface) that passes from point to point along a generatrix will change its direction. As the normal

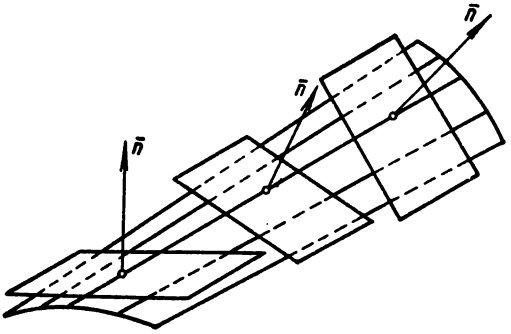


Fig. 277

revolves, so will the tangent plane, and the axis of revolution will be that generatrix along which the point of tangency moves (Fig. 277).*

But if all normals along a given generatrix are parallel to one another, the tangent plane will remain fixed as the point of tangency

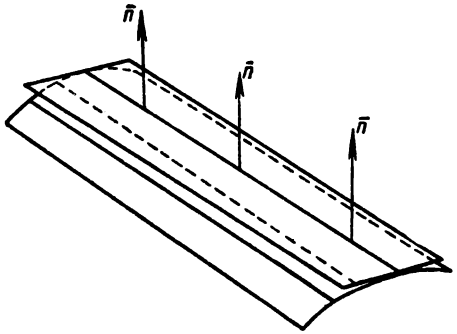


Fig. 278

moves along the generatrix (Fig. 278). When this condition is fulfilled for all generatrices, the ruled surface becomes developable (see Sec. 42).

We now consider the construction of a plane tangent to certain ruled surfaces.

* In Figs. 277 and 278 $\bar{n}$ denotes the vector of the normal.

Example 1. *To construct a plane tangent to a cone and passing through a point A that lies on its surface* (Fig. 279).

The generatrix SA drawn through the given point is the line of tangency. It will serve as one of the straight lines that define the desired plane P. The trace of plane P on plane R of the cone base (in a special case this may be the projection plane H or V) must pass through trace M of generating line SA.

Through point M lying in the cone base it is possible to draw a straight line that will either intersect the base or be tangent to it. This line will be the trace of an elementary cutting plane in the former case and a tangent plane in the latter.

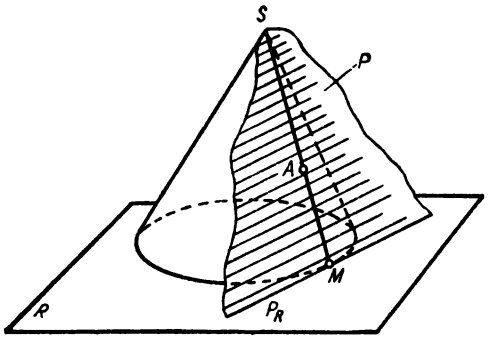


Fig. 279

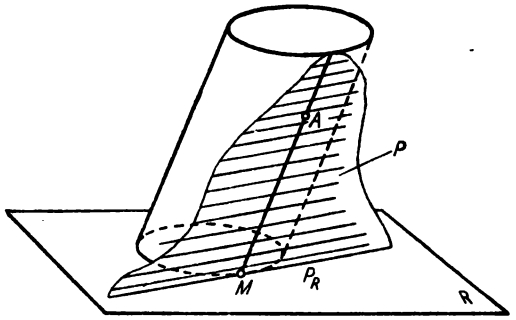


Fig. 280

Analogously (that is, by means of generatrix AM and trace P_R) we define a plane that is tangent to a cylinder and passes through point A given on its surface (Fig. 280).

When point A is given external to a conic surface (Fig. 281), the problem will have two solutions. Both planes P and Q will pass through a straight line connecting the cone vertex S with the given point A; on plane R their traces will be tangents to the cone base which are drawn through the like trace M of line SA. In the case of a cylindrical surface, this same problem is solved with the aid of a straight line that may be drawn through the given point parallel to the generating lines of the cylinder (Fig. 282). If the cylindrical surface is regarded as a conic surface with a vertex at

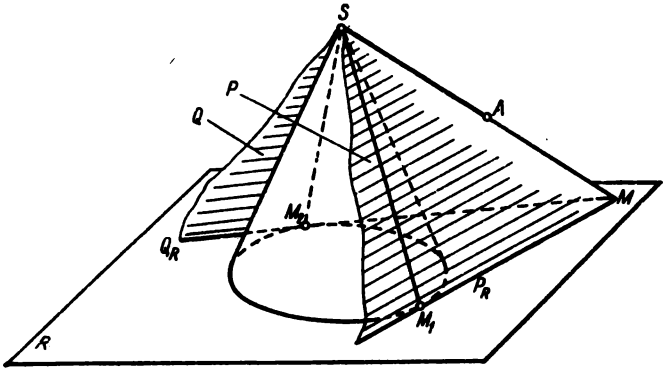


Fig. 281

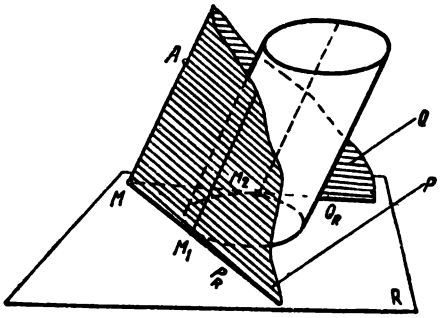


Fig. 282

infinity, the auxiliary straight line parallel to the generating lines of the cylinder will in no way differ from the straight line SA in Fig. 281.

Example 2. *Construct a plane tangent to a cone parallel to a given straight line AB.*

Any plane tangent to a cone must pass through its vertex S. But a further condition is imposed on the required plane, that of being parallel to the straight line AB. To satisfy both conditions, through vertex S draw a straight line SM parallel to AB (Fig. 283). We find trace M of this line and through the point obtained we draw straight lines tangent to the cone base. These will be horizontal traces of the required planes P and Q, which are tangent to the cone along the generating lines SK and SL. In Fig. 283, the vertical trace of line SM is used to construct P_V and Q_V.

The problem has a solution if trace M of the auxiliary line does not lie inside the base of the cone.

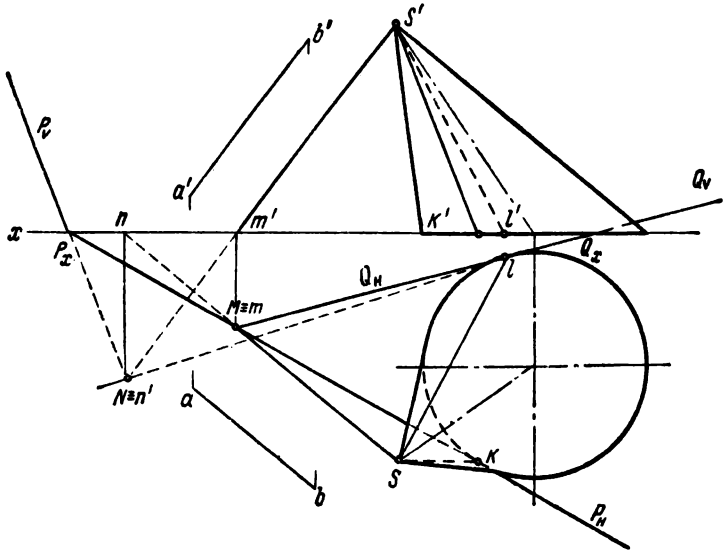


Fig. 283

The problem of constructing a plane tangent to a cone and parallel to a given straight line (as also the analogous problem of constructing a plane tangent to a cylinder) occurs when finding the shadows of solids. If these solids stand on a horizontal plane (the ground) it is convenient to use the horizontal traces of planes (see Fig. 283).

The problem is somewhat more complicated when the base of a body is located in some oblique plane (Fig. 284). To find the generatrices of tangency of planes parallel to a given straight line EF, draw through the cone vertex S a straight line SK parallel to EF.

Through SK draw a horizontal projecting plane Q (see trace Q_H), then find two points (G and M) of the line of intersection of planes Q and the cone base. Point I is the intersection of straight line SK with the plane of the cone base (see Sec. 23). Drawing from point I tangents IN and IL to the cone base, we find the points of tangency N and L. The two planes (parallel to EF) are tangent to the surface of the cone along the generatrices SN and SL.

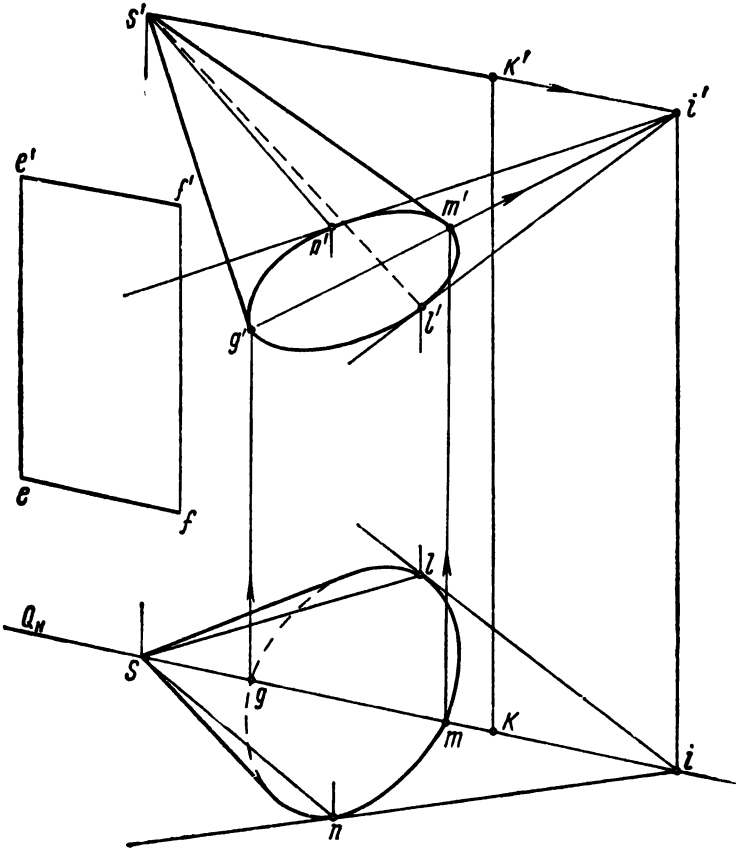


Fig. 284

Example 3. *Construct a plane tangent to a cylinder and parallel to a given straight line AB* (Fig. 285).

Since the required plane must contain one of the generatrices of the cylinder (the line of tangency) and a straight line parallel

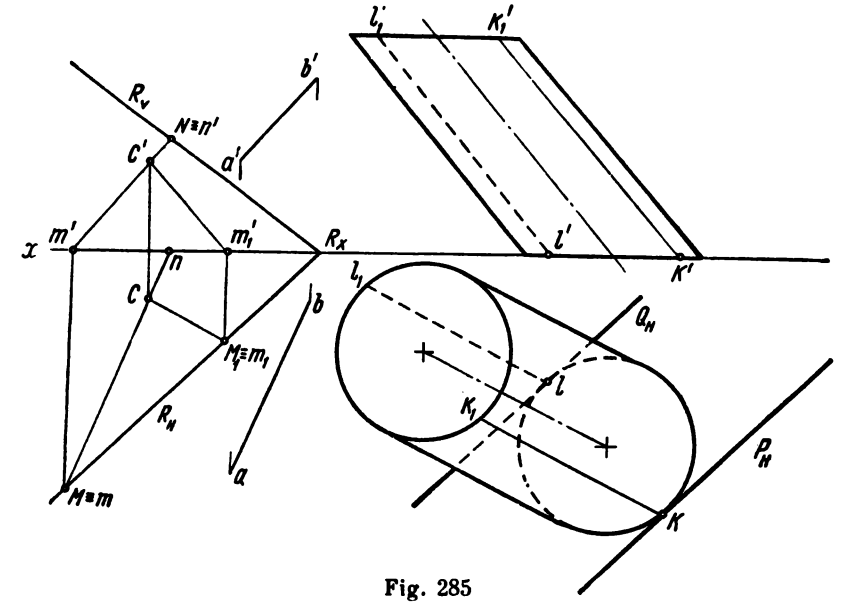


Fig. 285

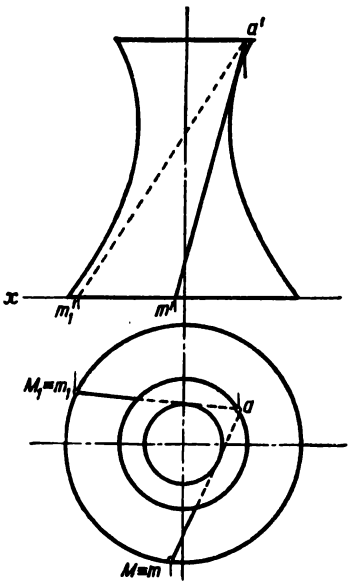


Fig. 286

to the given line AB, it is necessary, when determining the directions of the traces of the tangent planes, to construct a plane R parallel to AB and the generatrices of the cylinder. In Fig. 285, plane R is defined by intersecting lines: CM and CM_1 (CM is parallel to AB, CM_1 is parallel to the generating lines).

The two required planes P and Q will be parallel to R and will be tangent to the cylinder along the generatrices KK_1 and LL_1 respectively.

Example 4. *To construct a plane that is tangent to a hyperboloid of revolution of one nappe and passes through a point A on its surface.*

A hyperboloid of revolution of one nappe is a doubly ruled surface. Through each point of this surface it is possible to draw two straight-line generatrices, which will define the desired plane.

Being tangent to the surface at the given point, this plane will intersect the hyperboloid along two straight lines. In Fig. 286 the horizontal projections of straight-line generatrices are constructed as tangent lines to the gorge circle drawn from a. The vertical projections of these straight lines are obtained with the aid of points M and M_1, in which the generatrices intersect the lower base of the hyperboloid.

It will be noted that a plane tangent to another doubly ruled surface (the hyperbolic paraboloid) is also determined by the two straight-line generatrices that pass through a given point on the surface.

Sec. 49. Planes Tangent to Double-curved Surfaces

When constructing a plane tangent to a double-curved surface it is necessary on a surface to draw through a given point two curves, the tangents to which will determine the desired plane.

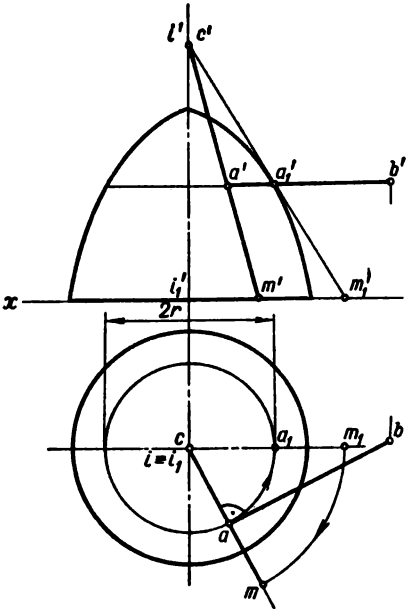


Fig. 287

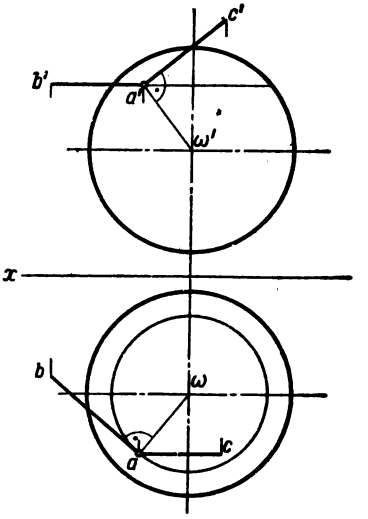


Fig. 288

Example 1. *To construct a plane tangent to the surface of revolution at a given point A on the surface* (Fig. 287).

If a single projection of the point is given (say, a'), the other projection may be found by means of the following auxiliary constructions. Through the given point on the surface draw a parallel circle of radius r. The vertical projection of this parallel will be a horizontal line-segment passing through a'. The same parallel will be projected on the H plane in the form of a circle. Using a line of recall we find a (in the given case it is assumed that point A is situated on the front half of the surface).

For curves passing over the surface through point A it is advisable to take the constructed parallel and a meridian. The tangent to the former is the line AB. Being in the same horizontal plane as the parallel, AB is projected onto V into a straight line that is parallel to the x-axis, and onto H in the form of a tangent to the circle of radius r. To construct the second straight line (tangent to the meridian), turn the meridian about axis II_1 to coincidence with

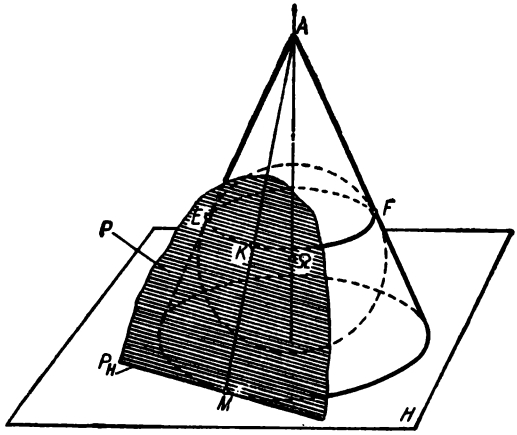


Fig. 289

the principal meridian. Then point A will occupy position A_1. Draw through A_1 a line tangent to the principal meridian and extend it to intersection with the axis I-I_1 at point C or to M_1 on plane H (one of these points will always be found within the limits of the drawing). It now remains to bring the meridian and the constructed tangent line to the original position. Joining point A with C or M, we get the second straight line, which intersects AB and determines the required tangent plane.

Example 2. *To construct a plane tangent to a sphere at a point A on its surface* (Fig. 288).

Since the radius of a sphere drawn to a point of tangency is a normal to the spherical surface, the problem of constructing a tangent plane reduces to the construction of a plane perpendicular to the radius ΩA. This plane may be defined by the straight lines AB and AC, the first of which is a horizontal line (ab is perpendicular to ωa), and the second is a vertical line ($a'c'$ is perpendicular to $\omega' a'$).

Example 3. *To construct a plane tangent to a sphere and passing through a given point A external to the sphere* (Fig. 289).

Through such a point it is possible to draw an infinity of straight lines tangent to the sphere. The locus of the tangent lines is a conic surface with vertex at a given point A. This conic surface is described about the sphere and is tangent along the circle $EKFE$. Any plane P that is tangent to the cone will also be tangent to the sphere. Indeed, plane P, which is tangent to the cone along the generatrix AM, and the sphere have only one common point K, the point of tangency. The problem thus allows for an infinitude of solutions. The required planes are easily constructed if a straight line joining point A and the centre of the sphere Ω is perpendicular to one of the projection planes. When $A\Omega$ is an oblique line, the orthographic representation has to be transformed so that one of the projections of $A\Omega$ becomes a point. The solution is completed by the construction of a plane tangent to an auxiliary right circular cone.

CHAPTER X

THE INTERSECTION OF CURVED SURFACES

Sec. 50. A General Method for the Construction of the Line of Intersection of Surfaces

To construct the points of the required line, cut the two given surfaces F_1 and F_2 with a third, Φ (Fig. 290), and construct the curves f_1 and f_2 along which the auxiliary surface Φ cuts F_1 and F_2.

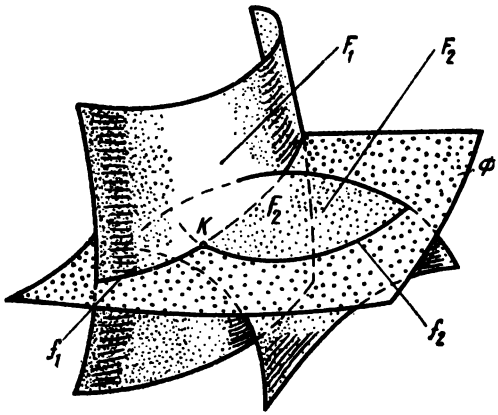


Fig. 290

The point (or points) of intersection K of these curved lines will belong to the required intersection line of the given surfaces. This line is often called a transition line. Repeating the foregoing constructions with a number of auxiliary surfaces, we determine the points of the curve that permit drawing it. Here, the auxiliary cutting surfaces should be selected so that the lines f_1 and f_2 are simple and convenient in construction (straight lines and circles). The choice of auxiliary surfaces is predetermined by the kind of surfaces given and by their relative positions. Planes and spheres are most frequently used.

Sec. 51. The Intersection of Ruled Surfaces

In constructing the transition lines of conic and cylindrical surfaces, it is best to take elementary cutting planes as auxiliary planes. These cut both cones and cylinders along the generating lines.

In the intersection of two cones (Fig. 291), the auxiliary cutting planes must pass through the straight line S_1S_2 that connects the vertices of the cones. Each of these planes that pass through S_1S_2 will cut the given cones or be tangent to their generatrices. The intersection points *I*, *II*, *III*, *IV* of the generatrices will belong to the required line. All cutting planes used to determine the points of the desired line form a pencil whose axis is the straight line S_1S_2. Through the horizontal trace *M* of this axis there will pass like traces of the elementary cutting planes.

When constructing the transition line of a cone and cylinder (Fig. 292), the auxiliary planes will be elementary cutting planes if they pass through the cone vertex parallel to the generatrices of the cylinder. The axis of the pencil (line *SM*) must also meet these conditions. Fig. 292 shows one of the planes of the pencil, plane *P*, whose horizontal trace P_H passes through point *M*. Points *I*, *II*, *III*, *IV* are found at the intersections of the corresponding generatrices.

Finally, in the intersection of two cylinders (Fig. 293), the auxiliary cutting planes must be parallel to the generatrices of both the first and second cylinders. The directions of the traces of elementary cutting planes are found with the aid of a plane director. This plane, *P*, is determined by two intersecting straight lines *KM* and KM_0, which are respectively parallel to the generatrices of the first and second cylinder. The elementary cutting planes must be parallel to *P*. One of them, P_1, is shown in Fig. 293.

If the given ruled surfaces are not conic or cylindrical, then to construct the required line one must find the points of intersection of straight-line generatrices of one surface with the other.

Sec. 52. The Intersection of Surfaces of Revolution

1. *The intersection of two coaxial surfaces of revolution* (Fig. 294). Let the meridian of one of the given surfaces be a curved line and the meridian of the other a straight line. The intersection point *K* of the meridians will describe a circle as they revolve about the common axis. This circle is the common parallel circle of the two surfaces of revolution and will be their line of intersection. The number of such parallels is equal to the number of the intersection points of the meridians.

2. *The intersection of surfaces of revolution whose axes have a common point.* Fig. 295 depicts the projections of surfaces F_1 and F_2

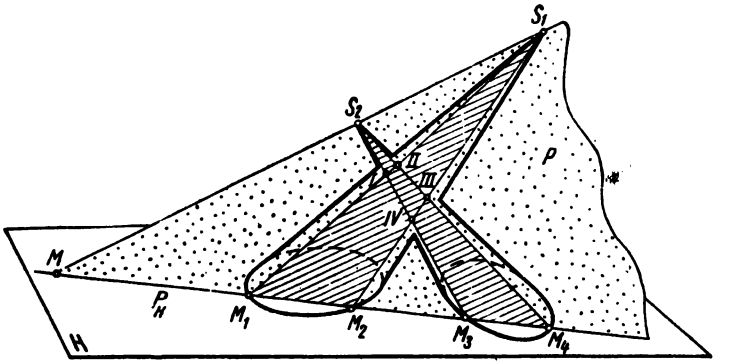


Fig. 291

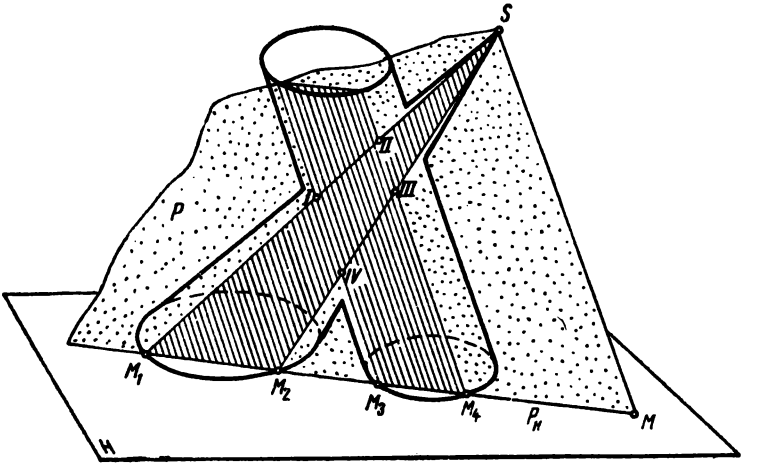


Fig. 292

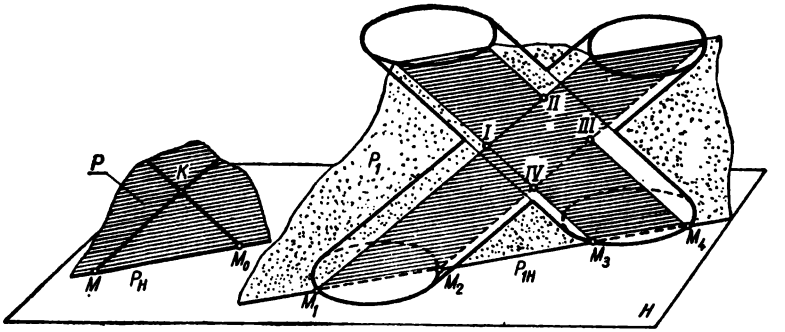


Fig. 293

on the V plane parallel to their axes. Here, for auxiliary planes we use spheres with common centre at the intersection point of the axes (point Ω). In this case, each of the auxiliary spheres, being coaxial both with F_1 and F_2, will cut the given surfaces in circles, whose projections on V will be straight lines. In Fig. 295, the straight lines $a_1'b_1'$ and $c_1'd_1'$ are the vertical projections of those parallel circles

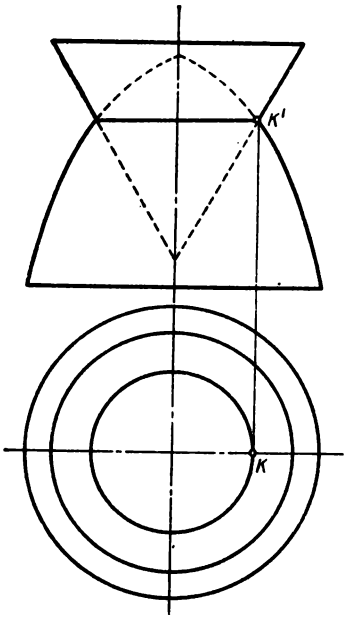


Fig. 294

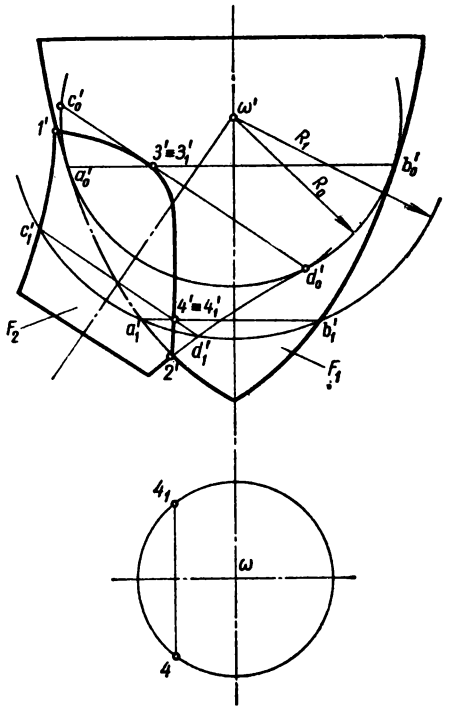


Fig. 295

in which the sphere of radius R_1 intersects F_1 and F_2. Intersection points $4'$ and $4_1'$ of the projections of the constructed parallels belong to the projection of the desired line. In a similar way we find points $3'$ and $3_1'$. Note however that straight line $a_0'b_0'$ is the projection on V of the line of tangency of surface F_1 and a sphere of radius R_0. The intersection of the principal meridians determines the extremities $1'$ and $2'$. Fig. 295 shows how horizontal projections of the parallels of surface F_1 are used to construct the horizontal projections of the points found (see the horizontal projections of points IV and IV_1).

Fig. 296 illustrates the intersection of ruled surfaces of revolution whose axes are in a single plane. Here the auxiliary surfaces are spheres, each one of which intersects the given surfaces of a cone and cylinders in circles (parallels). In Fig. 296 the projections of the circles have the following labels: $a_0'b_0'$ are the projections of two circles, along which two spheres of radius R_0 are tangent to a horizontal cylinder; $a_1'b_1'$ and $a_2'b_2'$ are the projections of parallels along

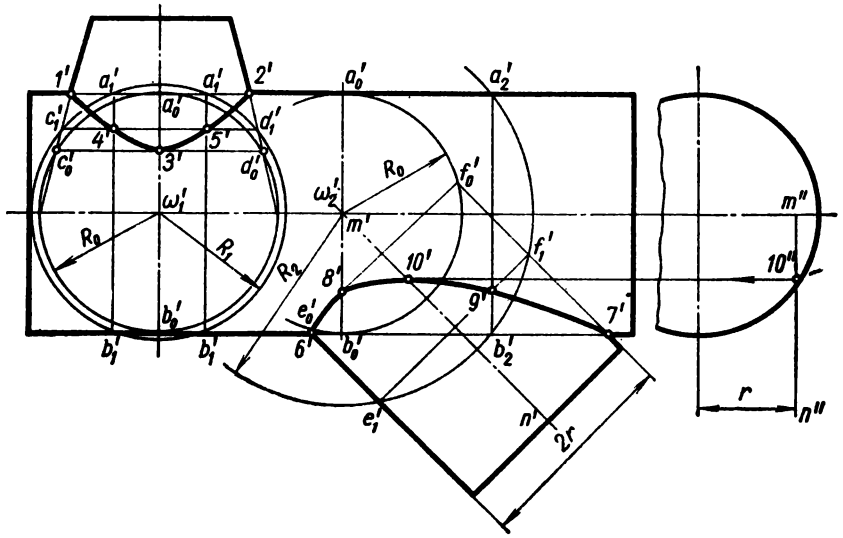


Fig. 296

which the auxiliary spheres of radii R_1 and R_2 intersect the same cylinder; $c_0'd_0'$, $c_1'd_1'$, $e_0'f_0'$ and $e_1'f_1'$ are the projections of circles in which the spheres intersect the cone and an oblique cylinder. These straight lines intersect in pairs and define the points of the vertical projections of the required lines. To construct the uppermost point *10'* of the intersection line of the two cylinders we had to take advantage of the profile projections of the horizontal cylinder and the generatrix *MN* of the oblique cylinder.

The intersection lines of the solids depicted in Fig. 296 might be constructed differently by using horizontal planes for one pair of solids and vertical planes (parallel to *V*) for the other. To each point found (with the exception of points *1'*, *2'*, *6'*, and *7'*) there corresponds a second point on the hidden side of the cylinders and cone.

The method of auxiliary spherical surfaces may also be employed when one of the given surfaces of revolution is a sphere.

Fig. 297 illustrates the projections of a part of a sphere of radius R and a surface of revolution with axis II_1 on plane V, which in this case is parallel to a plane defined by the centre of sphere C and axis II_1. Since any two spheres are always coaxial, the centre of the auxiliary spherical surfaces may lie in an arbitrary point Ω of axis II_1.

The auxiliary sphere of radius ρ intersects the given surfaces in circles whose projections on V are straight lines $a'b'$ and $e'f'$. Points III and III_1, the points of intersection of these circles, will belong to the given surfaces and, hence, to the required line.

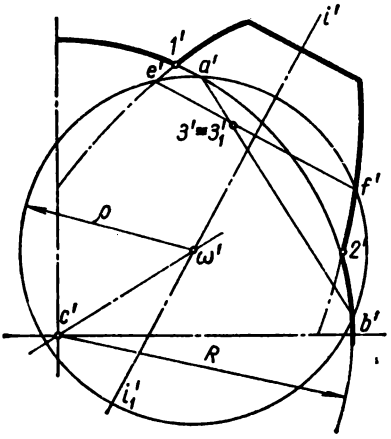


Fig. 297

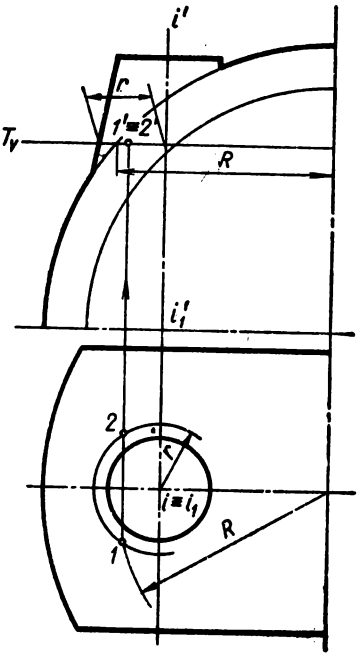


Fig. 298

If axis II_1 of the surface of revolution cut by the given sphere is perpendicular to one of the projection planes (Fig. 298), then planes perpendicular to the axis II_1 may serve as the auxiliary surfaces, One of these planes, T, is shown in Fig. 298. Plane T cuts both cone and sphere in circles of radii r and R. Both circles are projected on H true size. Their intersection points (I and II) belong to the required line. This method is repeatedly employed in the construction of the transition line of a taper boss and the spherical surface of a bearing cap (Fig. 299).

The intersection of a sphere with a cylinder is shown in Fig. 300. Here, we first find the reference points *I* and *II* on the contour generatrices of the cylinder. To do this, through the cylinder axis pass an auxiliary plane U_1 parallel to *V*. Plane U_1 intersects the

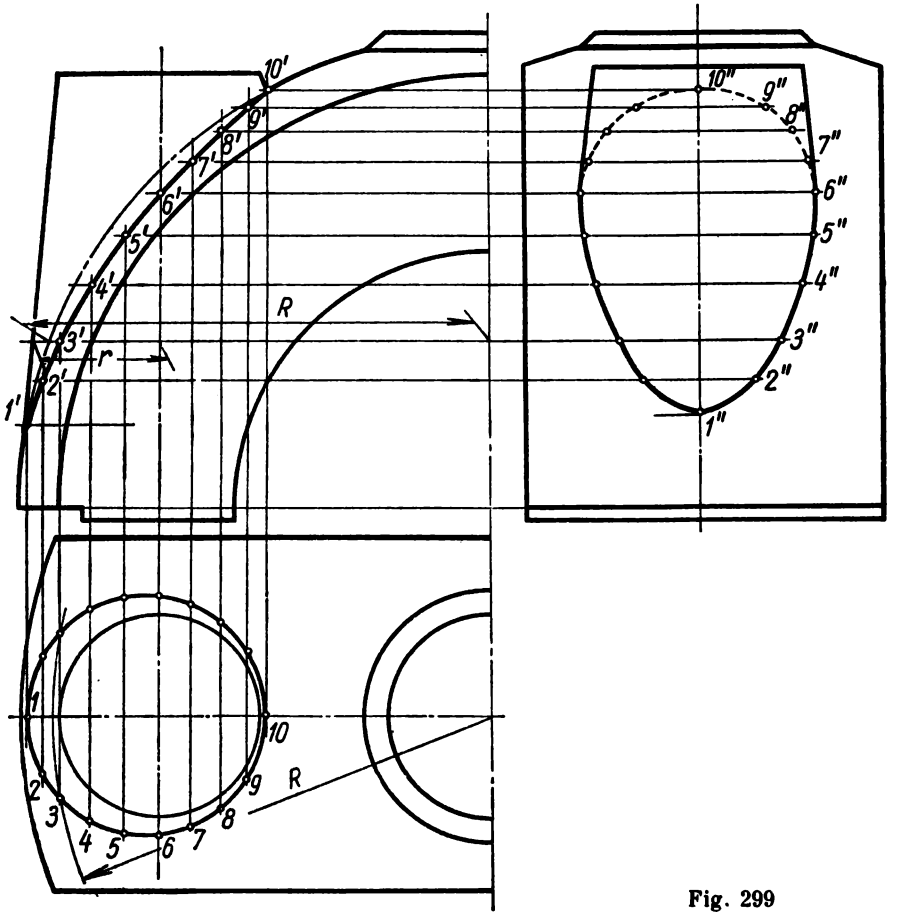


Fig. 299

cylinder along the contour generatrices, and the sphere, in a circle of radius ρ_1. The latter is projected on *V* true size and determines points *1'* and *2'* as it intersects the projections of the extreme generatrices of the cylinder.

Then we find points *III* and *IV* on the principal meridian of the sphere. The projections of the given surfaces are transformed in order to construct the upper and lower points (*V* and *VI*).

The new plane V_1 is parallel to P, the plane of symmetry of the two given surfaces. It is precisely in this plane P that we should find the self-symmetrical uppermost and lowermost points. Intermediate points of the transition line are obtained with the aid of planes parallel to V.

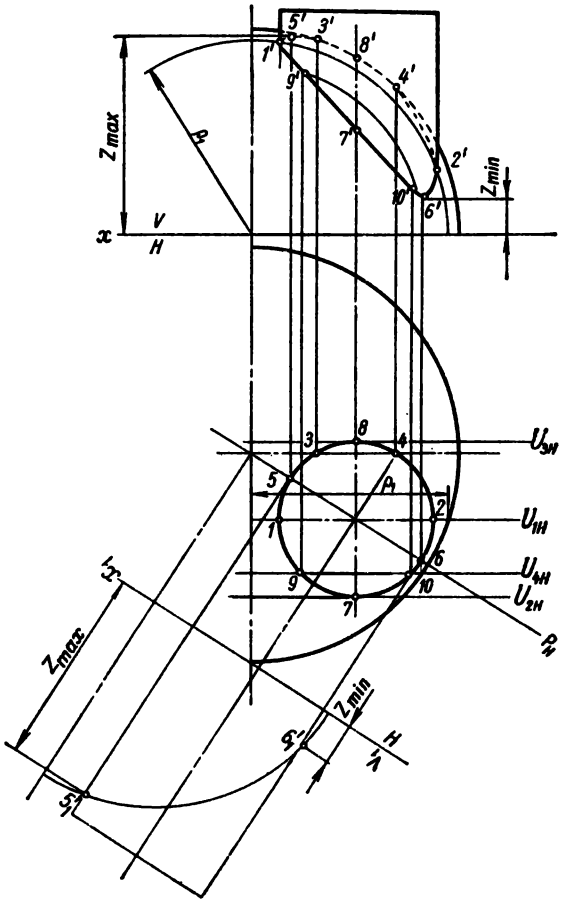


Fig. 300

3. *The intersection of surfaces of revolution whose axes do not lie in a single plane.* In this case the choice of auxiliary surfaces and their orientation relative to the projection planes depend on the statement of the concrete problem. For example, if the axis of a

double-curved surface of revolution is perpendicular to H and the cylinder axis crossing it is horizontal, we should take the horizontal planes T_1, T_2, and T_3 as the auxiliary surfaces (Fig. 301). Each of them intersects the first surface in parallels (circles) and the second along straight lines (generatrices). The points of intersection of the circles and of straight lines lying in one plane determine the required transition line.

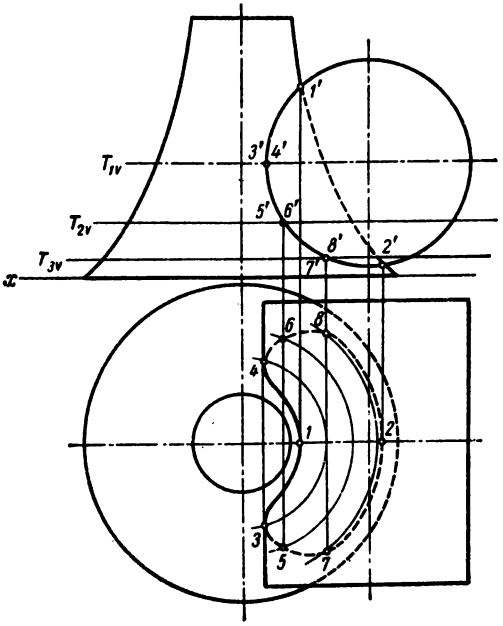


Fig. 301

4. Fig. 302 shows the intersection of a trihedral prism with a sphere of radius R.

The upper base of the prism is cut by the sphere along the arc DE of a circle of radius R_1.

The lateral face BC is cut along the arc DIF, and the lateral face CA along the arc FKE. The points which define the lines of intersection of the prism and sphere are found by means of auxiliary horizontal planes (for instance, points K and I by plane T).

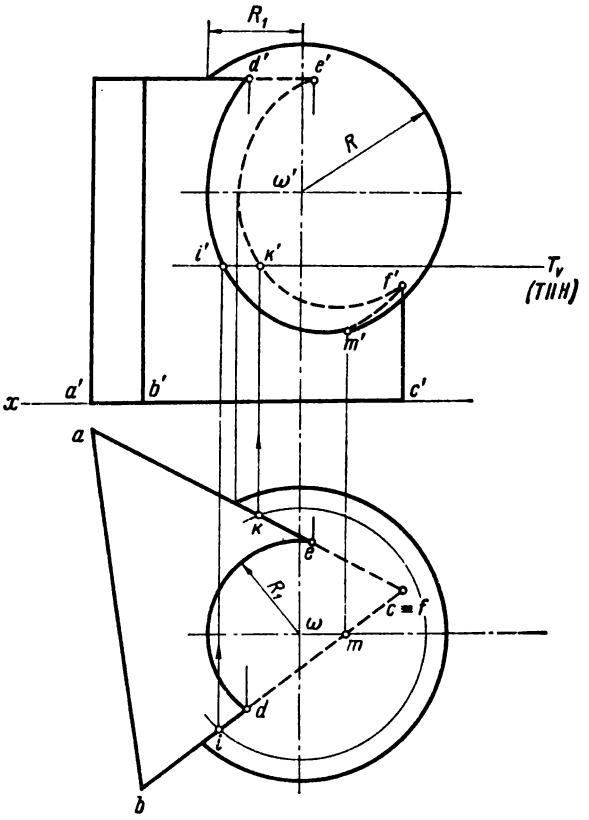


Fig. 302

Sec. 53. A Special Case of the Intersection of Quadric Surfaces

A quadric surface is a surface defined in a Cartesian coordinate system by an algebraic equation of the second degree. Quadric surfaces include: the sphere, ellipsoid, paraboloid, hyperboloid (of one nappe and of two nappes), the hyperbolic paraboloid, cones and cylinders whose generatrices are second-degree curves.

A frequent practical case is when two intersecting quadric surfaces are tangent to a third surface, which is also a quadric surface. When constructing a transition line in such a case, it is well to make use of the Monge theorem, which states that two quadric surfaces

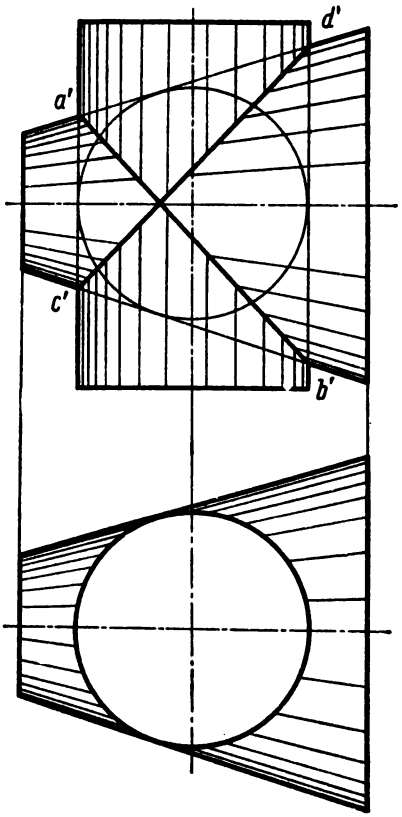


Fig. 303

intersect along two plane curves if they are circumscribed about a third surface or are inscribed in it.

Accordingly, the intersection lines of a cone and cylinder circumscribed about a sphere (Fig. 303) will be plane curves: ellipses, the vertical projections of which are depicted by the straight lines $a'b'$ and $c'd'$.

CHAPTER XI

CONSTRUCTION OF DEVELOPMENTS

Sec. 54. The Development of Polyhedrons

The development of a polyhedral surface is a plane figure obtained by bringing all the faces into coincidence in a single plane.

1. *The development of a pyramid.* The lateral faces of a pyramid are triangles, for the construction of which we need only determine the true lengths of their sides (the edges of the pyramid). Referring to Fig. 304, we see that the base of the pyramid is situated in the H plane; hence, it is sufficient to determine the lengths of the lateral edges, which is done by the method of revolution. The true lengths of the lateral edges are denoted by l_{SA}, l_{SB} and l_{SC}. Then, using three sides, construct the contour of one face, then a second adjacent face, etc. The development is completed by constructing the base of the pyramid (triangle ABC), which may be affixed to any one of the faces. Construction is greatly simplified if we have a right pyramid. Here, all lateral faces are equal isosceles triangles. In certain cases it is necessary to indicate points that belong to the surface of the given pyramid. This kind of problem arises when it is required to construct, on the development, a line along which the pyramid is cut by a plane or by another polyhedron, or by a curved surface.

Let there be given a point K lying on face SAC. We shall show how this point is developed. Through K draw in the face a straight line joining K with the pyramid vertex S. To plot this line on the development we must know the positions of two points of the line on the development. These points are the pyramid vertex S and point M located on AC, an edge of the base. On the development, point M is constructed by means of a line-segment of length l. We now have to find the distance from K to vertex S (line-segment l_{SK}) and lay it off on the development from S along the auxiliary straight line SM. It is somewhat easier to determine the points of a development that belong to the edges, that is, when $l=0$ (for instance, point N).

2. *The development of a prism.* Since the faces of a prism are parallelograms, we shall first determine the true size of this figure. By the methods of descriptive geometry, the projections of any parallelogram may be transformed as shown in Fig. 305, when two

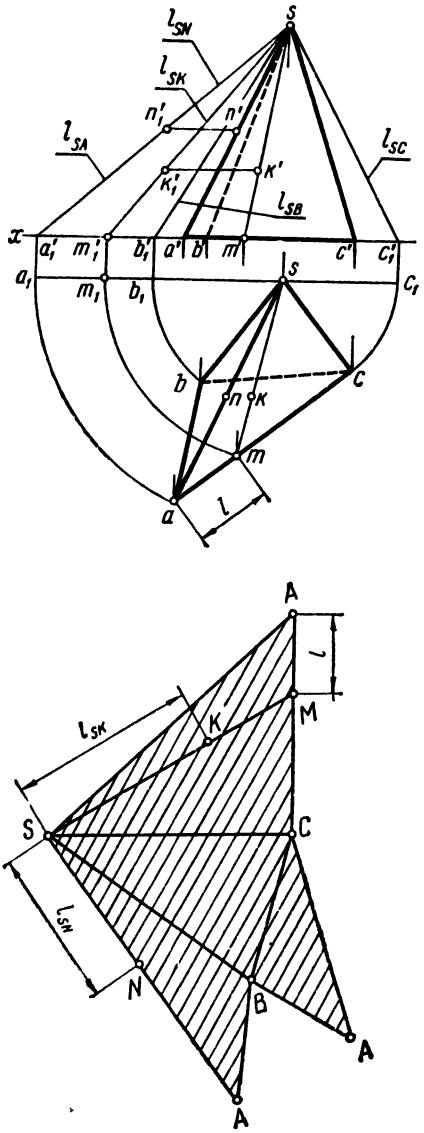


Fig. 304

opposite sides (*AB* and *CD*) are parallel to *H* and the other two (*AD* and *BC*) are parallel to *V*.

In such a case, the parallelogram may be turned about any one of its sides to a position parallel to the projection plane to which the axis of revolution is parallel. The *V*-parallel *AD* is the axis of revolution in Fig. 305. As the parallelogram is revolved, points b' and c' move (in the orthographic representation) along lines perpendicular to the axis of revolution or to the straight line $a'd'$. At the instant that the plane of the figure is parallel to *V*, all its sides project on *V* true size.

Consequently, the vertex *B* must be displaced from the fixed point a' by a distance l_{ab}, which is taken from the horizontal projection. The same is true for the relative positions of points *C* and d'. Thus, striking arcs of radius l_{ab} on perpendiculars along which b' and c' move, we get points *B* and *C* of the contour that defines the true size of the parallelogram. We refer to Fig. 306, where the projections of an oblique triangular prism were given in the *V*/*H* system. To construct the development we had to transform the orthographic representation by building a new vertical projection of the prism on plane V_1, which is parallel to the lateral edges of the given polyhedron.

The development of the lateral surface of the prism is begun with face *ABFE*, the construction of which is similar to what was done in Fig. 305. We then revolve the next face, *BCGF*, around edge *BF*.

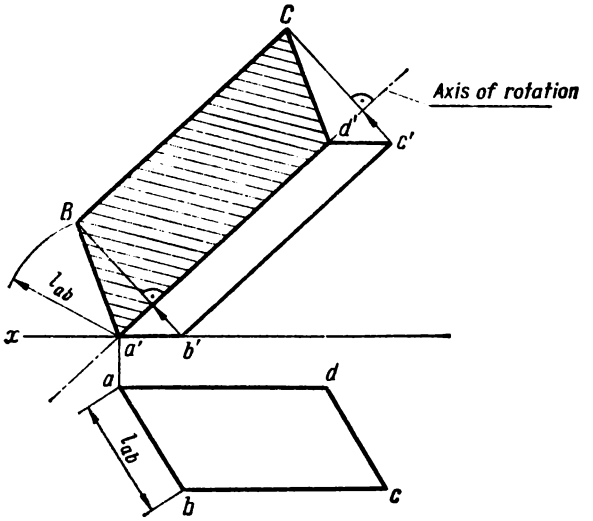


Fig. 305

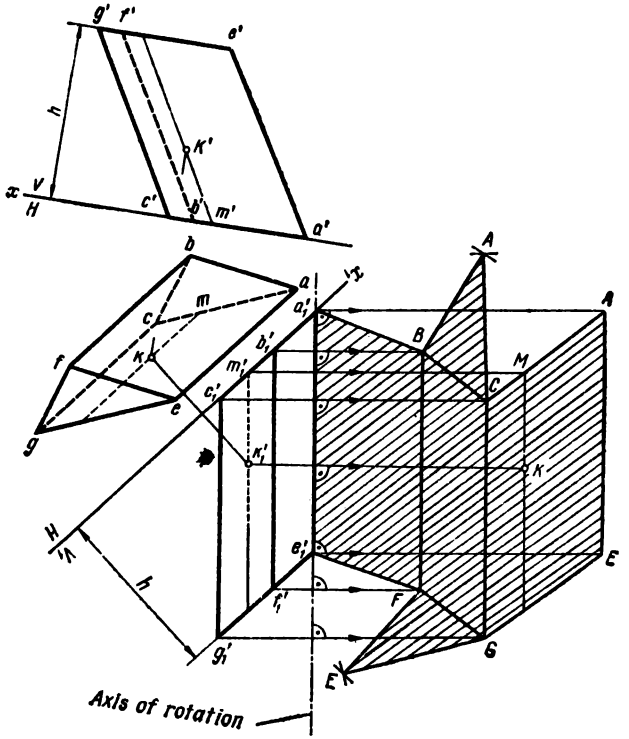


Fig. 306

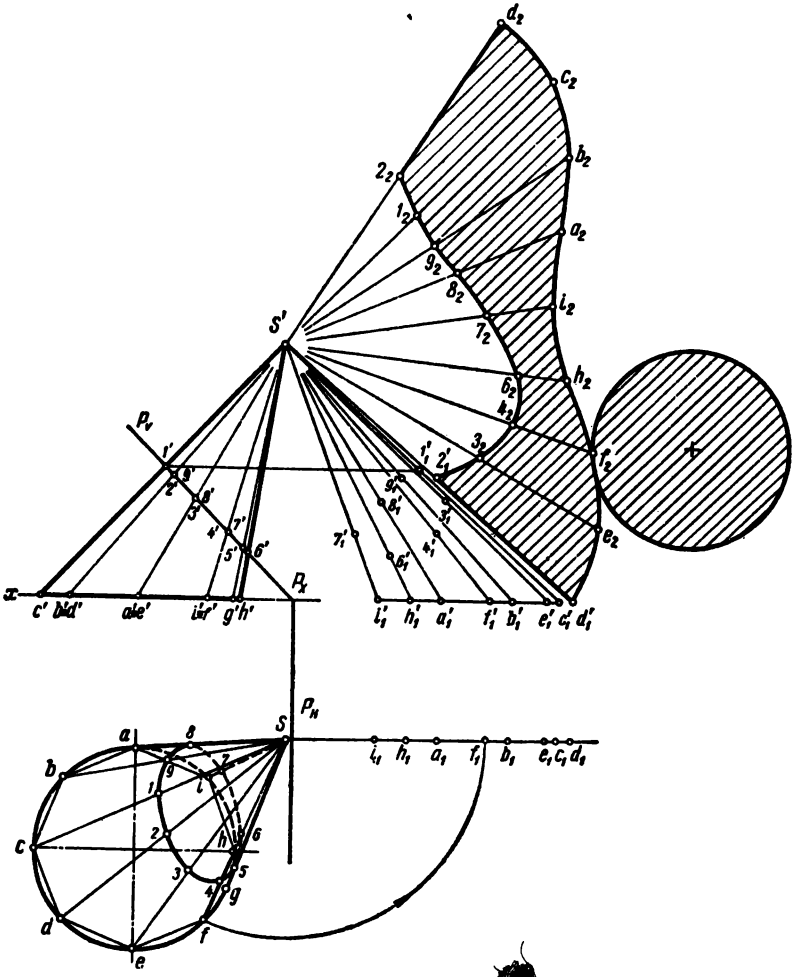


Fig. 307

On the perpendiculars (to the AE axis) along which points c_1' and g_1' move, we strike arcs of radius bc from points B and F as centres. This yields vertices C and G on the development. The procedure is repeated as many times as there are lateral faces. Fig. 306 also shows how to determine on a development the position of some point K belonging to face $ACGE$. Through a given point draw in the face a straight line KM parallel to the lateral edges of the prism. First construct an auxiliary line on the development and then the given point K.

Sec. 55. The Developments of Cones and Cylinders

Here we do not consider right circular cones and cylinders, the development of which is studied in secondary-school geometry. To develop oblique and right (but not circular) cones and cylinders, do as follows:

1. Inscribe a many-sided (n-gonal) pyramid or prism in the given cone or cylinder. The number n depends on the size of the drawing, but it should never be taken less than six.

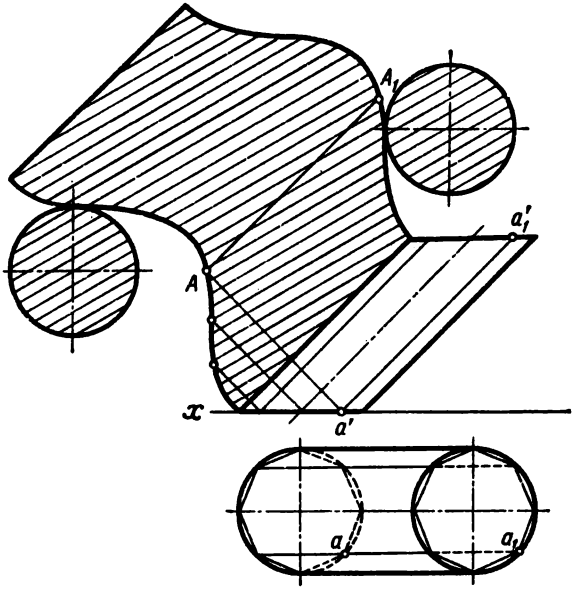


Fig. 308

2. Develop the n-gonal pyramid or prism as described in Sec. 54.
3. Connect the edges of the development with smooth curves.

We thus have the developments of a cone and cylinder as shown in Figs. 307 and 308. In the development of the cone, also indicated is its line of intersection with plane P.

Sec. 56. The Development of the Sphere

A spherical surface is undevelopable. Existing methods for constructing its development yield only approximate results. In one of the methods, an element of the spherical surface is replaced by an element of a cylindrical surface. Here, the element of the sphere is that part of it which is bounded by two great circles (called a gore).

Given a sphere (Fig. 309), the surface of which is divided by great circles into several (say eight) equal parts. Each of these elements of the sphere is projected on the H plane in the form of a sector. Now, about the sphere describe a cylindrical surface, the axis of which passes through the centre of the sphere perpendicular to V (in Fig. 309 the contour of the cylindrical surface is indicated by a dash-and-dot line).

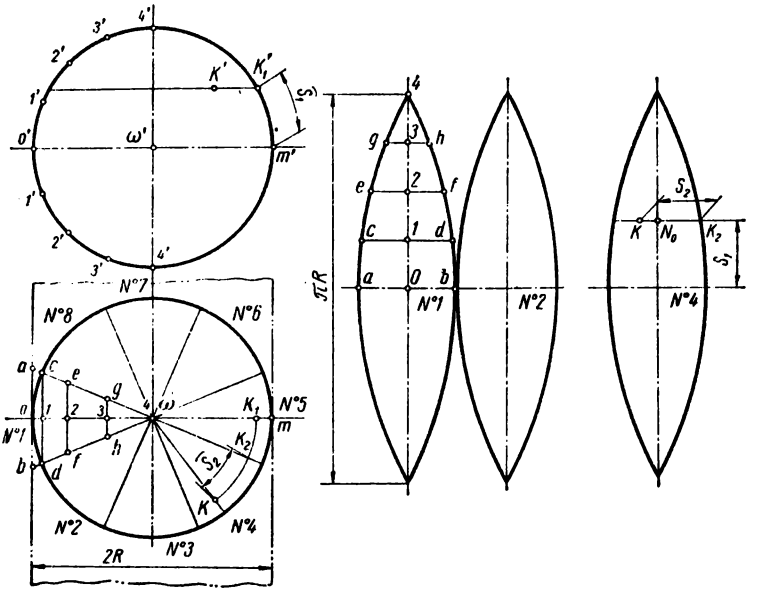


Fig. 309

To an accuracy sufficient for practical purposes, we replace an element of the sphere with part of the surface of the constructed cylinder. Triangle $ab\omega$ is the horizontal projection of this cylindrical element, the contour of the sphere (an arc of a circle) is the vertical projection. To construct the development of a cylindrical element (gore) we divide its vertical projection into eight equal parts and construct the horizontal projections of the generatrices that correspond to the division points. Since the generatrices of a cylindrical surface are parallel to H, the true lengths of the line-segments of the generatrices for constructing the development are taken from the horizontal projection (line-segments ab, cd, ef and gh), and the distance between them is measured on the vertical projection (the distances between points $0'$ and $1'$, $1'$ and $2'$, etc.).

The development of three (out of eight) gores of the surface of the sphere is shown in Fig. 309 to the right. Through the midpoint of *ab* is drawn a vertical axis of symmetry of a gore. On this axis we lay off four segments *0-1*, *1-2*, etc., upwards and downwards. Through points *1*, *2*, and *3* are drawn horizontal lines on which we lay off segments *cd*, *ef* and *gh*.

Joining the extremities of the segments with a smooth curve, we get the development of the upper half of a gore. The lower half is analogously constructed. Eight such gores yield an approximate development of the sphere.

The position of point K on the sphere is defined on the development by two "coordinates": arcs S_1 and S_2.

Arc S_1 indicates the displacement of point K from the equator to a pole. Arc S_1 is equal to that part of a sphere meridian bounded by the equator and a parallel passing through point K. The length of this arc $S_1=k_1'm_1'$ is laid off on the development from the equator to the appropriate gore (in our case, No. 4) along the vertical axis of symmetry. Through the point thus obtained, N_0, draw a horizontal straight line; it is the parallel that contains K.

It remains to lay off on this parallel from the right contour of the gore a line-segment K_2K, equal to arc S_2. Arc S_2 is depicted true size on the horizontal projection of the sphere. It is part of the parallel that passes through point K (from K to the contour of gore No. 4).

Part two

PROJECTIONS: AXONOMETRIC, PERSPECTIVE AND WITH ELEVATIONS

*

CHAPTER XII

THE METHOD OF PARALLEL PROJECTION ON A SINGLE PLANE (AXONOMETRY)

Sec. 57. The Fundamentals of the Method

Axonometric views are widely used due to their pictorial nature and the simplicity of construction.

Axonometry means measuring by means of axes. The axonometric method may be combined both with parallel and central projection, provided the object is projected together with a coordinate system.

Essentially, the method of parallel axonometric projection consists in the fact that an object is referred to some coordinate system and then is projected by parallel lines or rays onto a plane together with the system of coordinates.

Fig. 310 shows a point A of space referred to a system of rectangular coordinates $Oxyz$.

The line-segment MN defines the direction of projection on the picture plane K (the projection plane).

To produce an axonometric* (in our case, parallel) projection of a point A, draw through the point a projecting line (parallel to MN)

* From now on we shall call a parallel projection axonometric, bearing in mind, however, that an axonometric projection may also be a central projection.

and find its intersection with plane K at point A_K. This construction shows that for a given direction of projection there is a specific point A_K that corresponds on the projection plane to each point A of space.

The converse, however, cannot be asserted. Indeed, to every point A_K in plane K there will correspond any point of the projecting line A_KA.

To eliminate this indeterminacy and ensure a one-to-one correspondence between the points of space and the points of the picture

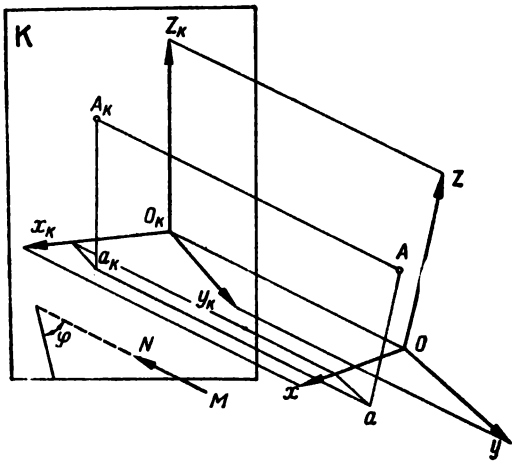


Fig. 310

plane, do as follows: project on plane K not only point A but also one of its orthogonal projections. It is common to take the horizontal projection a on the coordinate plane xOy.

The term secondary projection is applied to the axonometric projection a_K of the horizontal projection of point A. This term clearly expresses the fact that point a_K is obtained by means of two successive projections.

Again examining Fig. 310 we conclude that if we start with a coordinate system $Oxyz$, a direction of projection MN and plane K, then the axonometric projection of the point and its secondary projection uniquely define the position of the point in space. Indeed, drawing through the secondary projection a_K of point A a straight line parallel to MN and then determining the point of intersection of this line with the coordinate plane xOy, we find the horizontal projection a of A. Now the position of A in space will be determined by the intersection of two straight lines A_KA and aA,

the first of which passes through A_K parallel to MN and the second through a, perpendicular to the xy-plane.

Plane K (see Fig. 310) also shows the axonometric projection of the coordinate axes: the plane system $O_K x_K y_K z_K$. In the general case the lengths of the line-segments of the coordinate axes in space will not be equal to the lengths of their projections.

Distortion of the segments of the coordinate axes during projection on plane K is characterized by so-called distortion factors (ratios of foreshortening, or scale ratios). The distortion factor is the ratio of the length of the projection of an axis on the picture to its true length. Thus, the distortion factor along the axis $O_K x_K$ is $s=\frac{O_K x_K}{Ox}$ along the axis $O_K y_K$, $t=\frac{O_K y_K}{Oy}$, and along the axis $O_K z_K$, $u=\frac{O_K z_K}{Oz}$ (Fig. 310).

Depending on the ratios of distortion factors, axonometric projections may be:

(1) *isometric*, if the distortion factors are equal along all three axes. In this case $s=t=u$;

(2) *dimetric*, if the distortion factors are equal along any two axes and if the third differs from these two: say, $s=u\neq t$;

(3) *trimetric*, if all three distortion factors differ, i.e., $s\neq t\neq u$; $s\neq u$.

Axonometric projections likewise differ in the angle φ, which is formed by a projecting line with the projection plane K. If $\varphi\neq 90°$, the axonometric projection is oblique, and if $\varphi=90°$ it is rectangular.

Naturally, isometric, dimetric and trimetric projections may be either rectangular or oblique.

Sec. 58. Certain Properties of Parallel Projection

In Sec. 3 we established the properties of relative positions of points, lines and plane figures that are preserved under a parallel projection.

Starting with these properties of parallel projection and knowing the distortion factors, it is possible to construct an axonometric image of a point. This image is defined as the boundary point of a coordinate polygonal line consisting of line-segments of length x_K, y_K, z_K laid off from the origin of the axonometric axes O_K along appropriate straight lines parallel to these axes (Fig. 311) or coincident with them.

The construction of a coordinate polygonal line requires measuring three rectangular coordinates of the point, x, y, z, their conversion into axonometric coordinates with the aid of distortion factors, and, finally, drawing the polygonal line and, concomitantly, one of the secondary projections of the point.

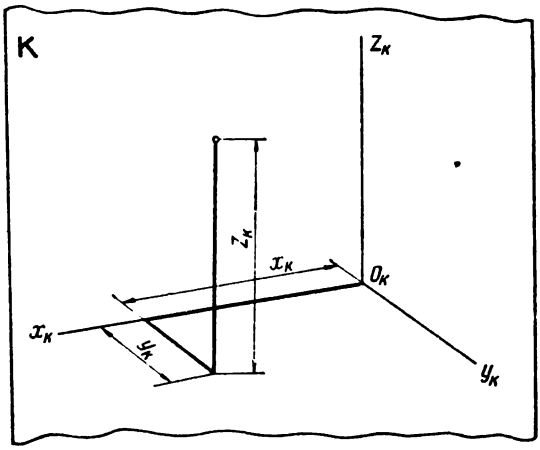


Fig. 311

Sec. 59. The Basic Theorem of Axonometry

When constructing a parallel projection, we may arbitrarily choose a projection plane K and the direction of projection.

Obviously, any change in the relative positions of the coordinate axes and the projection plane and any change in the direction of projection will give rise to changes both in the position of the axonometric axes and in the distortion factors along them.

In 1853, the geometer K. Polke, studying the question of the directions of the axonometric axes and the distortion factors along them as functions of the direction of projection and the position of the projection plane, arrived at the following conclusion (see Fig. 310).

Three arbitrarily chosen line-segments O_Kx_K, O_Ky_K, O_Kz_K on plane K emanating from a single point represent a parallel projection of three equal and mutually perpendicular line-segments Ox, Oy, Oz emanating from some point of space.

This theorem is of fundamental significance both for the theory of axonometry and for many of its applications. Applying Polke's theorem, it is possible to specify, in an arbitrary fashion, the axonometric axes and also the ratio of distortion factors along them.

The distortion factors will accordingly be proportional to segments depicting the axonometric axes. Indeed, the segments O_Kx_K, O_Ky_K and O_Kz_K, which are numerators of fractions that determine the distortion factors $\frac{O_Kx_K}{Ox}$, $\frac{O_Ky_K}{Oy}$, $\frac{O_Kz_K}{Oz}$, may, according to Polke's theorem, be chosen arbitrarily. But all these three arbitrarily chosen segments

serve as the parallel projections of three equal and mutually perpendicular segments of space; let the length of each of them be m. Forming the ratio $s : t : u = \frac{O_K x_K}{Ox} = \frac{O_K y_K}{Oy} = \frac{O_K z_K}{Oz}$ and substituting m for Ox, Oy, Oz we get

$$s:t:u = O_K x_K : O_K y_K : O_K z_K$$

which proves that the distortion factors are proportional to the corresponding line-segments.

Sec. 60. The Relationship Between Distortion Factors and the Angle of Projection

The following relationship exists between the distortion factors and angle φ which the direction of projection makes with plane K:

$$s^2 + t^2 + u^2 = 2 + \cot^2 \varphi$$

To prove this, let us examine Fig. 312, which depicts a rectangular coordinate system $Oxyz$ and a projection plane K. The direction of projection is given by the segment OO_K, point O_K being the projection of the origin O on plane K.

Segments O_KA, O_KB and O_KC are axonometric coordinate axes.

Angle φ, which is formed by the direction of projection and the projection plane K, is measured by angle OO_KD. Side O_KD of this angle serves as the orthogonal projection of the direction of projection OO_K on plane K. Point D is thus the foot of the perpendicular dropped from the origin O onto the projection plane K.

Let us label the angles formed by the direction of projection OO_K and the coordinate axes in space with the letters α, β, γ, and the angles formed by the perpendicular OD with the same axes with α_1, β_1, γ_1 (in Fig. 312, angles β_1 and γ_1 are not indicated).

From triangle OO_KA it follows that

$$O_KA^2 = OO_K^2 + OA^2 - 2 \cdot OO_K \cdot OA \cdot \cos \alpha$$

Dividing both sides by OA^2 and substituting s for $\frac{O_KA}{OA}$, we get

$$s^2 = \left(\frac{OO_K}{OA}\right)^2 + 1 - 2 \cdot \frac{OO_K}{OA} \cos \alpha$$

Now let us transform the ratio of the two segments OO_K and OA. Noting that in the right triangle ODO_K, $OO_K = \frac{OD}{\sin \varphi}$ and in the right triangle ODA, $OA = \frac{OD}{\cos \alpha_1}$, we get

$$\frac{OO_K}{OA} = \frac{\cos \alpha_1}{\sin \varphi}$$

After substituting this ratio in the formula for s^2, we have

$$s^2 = \frac{\cos^2 \alpha_1}{\sin^2 \varphi} + 1 - 2 \frac{\cos \alpha_1}{\sin \varphi} \cos \alpha$$

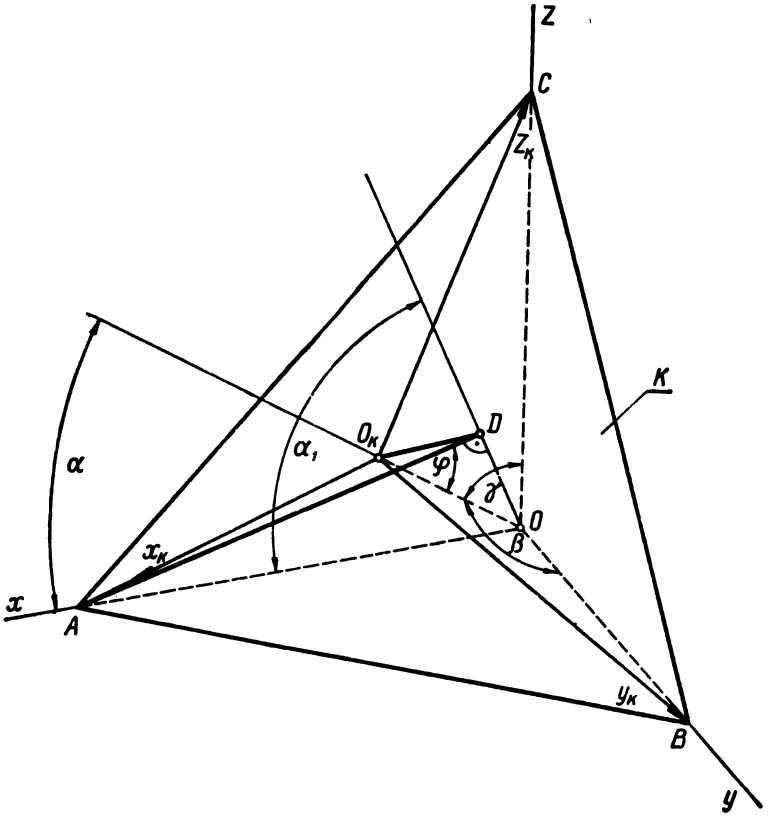


Fig. 312

Similarly, we obtain the formulas for distortion factors along the axes $O_K y_K$ and $O_K z_K$:

$$t^2 = \frac{\cos^2 \beta_1}{\sin^2 \varphi} + 1 - 2 \frac{\cos \beta_1}{\sin \varphi} \cos \beta$$

$$u^2 = \frac{\cos^2 \gamma_1}{\sin^2 \varphi} + 1 - 2 \frac{\cos \gamma_1}{\sin \varphi} \cos \gamma$$

After summing the left and right sides of these equations, we obtain

$$s^2+t^2+u^2=\frac{\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1}{\sin^2\varphi}+3-$$

$$-2\frac{\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1+\cos\gamma\cos\gamma_1}{\sin\varphi}$$

But since the sum of the squares of the cosines of the direction angles is equal to unity, and expression

$$\cos\alpha\cos\alpha_1+\cos\beta\cos\beta_1+\cos\gamma\cos\gamma_1$$

defines the cosine of the angle between segments OO_K and OD and is equal to

$\cos(90°-\varphi)=\sin\varphi$, it follows that

$$s^2+t^2+u^2=\frac{1}{\sin^2\varphi}+1$$

or, finally,

$$s^2+t^2+u^2=2+\cot^2\varphi \qquad (1)$$

In the case of rectangular axonometry, when $\varphi=90°$, the sum of the squares of the distortion factors will be equal to 2.

Sec. 61. Standard Axonometric Projections

The USSR Standard GOST 3453-59 recommends the use of rectangular isometry and dimetry. Above, it was shown that in rectangular axonometry the sum of the squares of the distortion factors is 2. But in isometry, $s=t=u$ and so $3s^2=2$, whence $s=\sqrt{\frac{2}{3}}\approx 0.82$.

Thus, in rectangular isometry the size of an object along all three dimensions is reduced by 18%. The Soviet Standard recommends that isometric projection should be constructed without foreshortening of the coordinate axes; this corresponds to an enlargement in the image by a factor of $\frac{1}{0.82}=1.22$.

In constructing a rectangular dimetric projection, foreshortening along the axis O_Ky_K is doubled as compared with the other two; that is, we take

$$s=u \text{ and } t=0.5s$$

Then by formula (1) we have

$$2s^2+(0.5s)^2=2 \qquad (2)$$

whence

$$s^2 = \frac{8}{9}$$

and

$$s \approx 0.94 \text{ and } t \approx 0.47$$

In actual constructions, the fractional factors are dispensed with by introducing a scale of enlargement defined by the relation $\frac{1}{0.94} =$

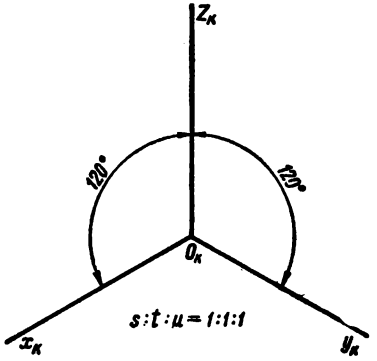


Fig. 313

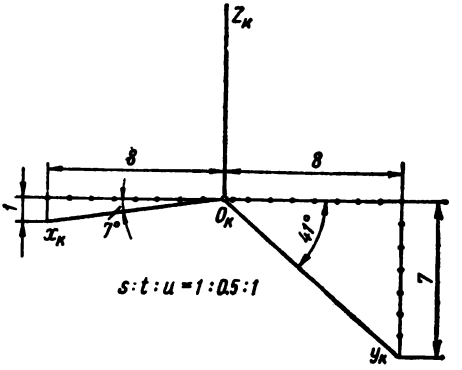


Fig. 314

$=1.06$; then the distortion factors along the axes $O_K x_K$ and $O_K z_K$ will be unity, and along the axis $O_K y_K$ one half: $t=0.5$.

The positions of the axes in rectangular isometry and dimetry are shown in Figs. 313 and 314, respectively.

Determining the angles between the axes shown in these figures is associated with solving simple stereometric problems, which we do not consider. Of the oblique axonometric projections, the Soviet Standard recommends the use of vertical (frontal) dimetry.

In this particular case, the projection plane K is situated frontal-parallel to the coordinate plane xOz (Fig. 315). Then the x- and z-axes, which are parallel to the K plane, are projected on it true size and, consequently, the distortion

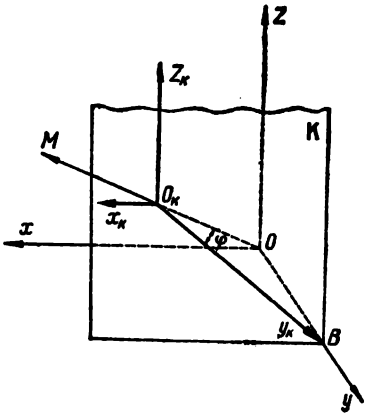


Fig. 315

factors along them will be equal to unity:

$$s = u = 1$$

The distortion factor along $O_K y_K$ is determined from the right triangle OBO_K by the relationship

$$t = \frac{O_K B}{OB} = \cot \varphi$$

where φ is the angle between the direction of projection and plane K.

Thus, oblique axonometric projections on a plane parallel to one of the projection planes are dimetric projections. But when $\varphi = 45°$

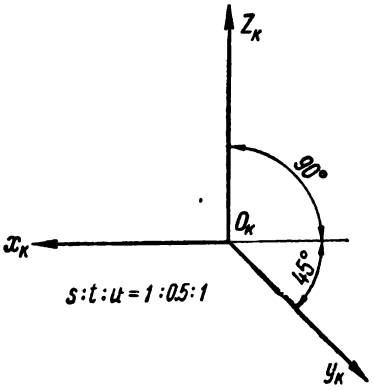


Fig. 316

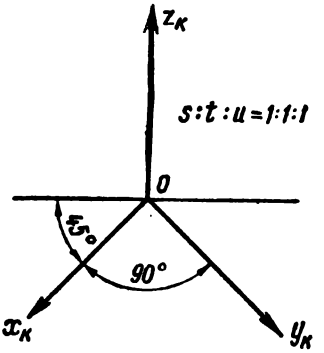


Fig. 317

(cot $\varphi = 1$) we have an oblique isometric projection with distortion factors equal to unity.

The Soviet Standard recommends constructing vertical (frontal) dimetry with foreshortening along the axis $O_K y_K$ by a factor of two (Fig. 316).

Employ vertical dimetric projections when it is advisable to retain true size figures located in planes parallel to the vertical projection plane. When the aim is to retain true size figures located in horizontal planes, the picture plane (K) is placed parallel to the H plane and all distortion factors are taken as unity. This type of oblique isometric projection is called "military perspective" (Fig. 317).

Sec. 62. The Circle in Axonometry

In the parallel projection of a circle on some plane K (Fig. 318), we get an image in the form of an ellipse (in the general case).

No matter how the plane of the circle is situated, first construct a parallelogram $A_K B_K C_K D_K$ (the parallel projection of the square $ABCD$

circumscribed about the given circle) and then with the aid of 8 points and 8 tangent lines, inscribe an ellipse in it.

The first four of these points (1_K, 2_K, 3_K, 4_K) are the midpoints of the sides of the parallelogram, the second four (E_K, F_K, G_K, H_K) are points located on the diagonals so that each of them divides the half-diagonal in the ratio 7 : 10.

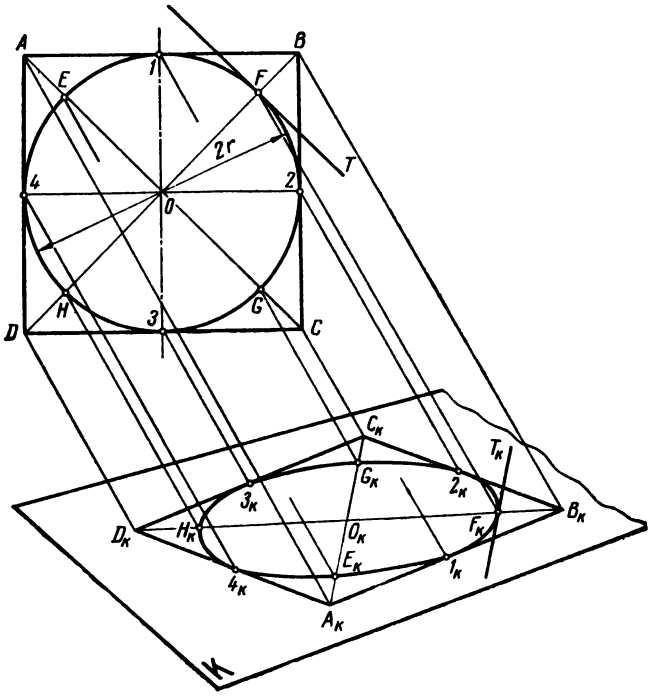


Fig. 318

Indeed, on the basis of the properties of parallel projection we can write that $\frac{OE}{OA} = \frac{O_KE_K}{O_KA_K}$.

But $\frac{OE}{OA} = \frac{r}{r\sqrt{2}} \cong \frac{7}{10}$ and hence we also have $\frac{O_KE_K}{O_KA_K} = \frac{7}{10}$.

For tangents at the points 1_K, 2_K, 3_K and 4_K we have the sides of the parallelogram, at the points E_K, F_K, G_K and H_K we have straight lines parallel to the corresponding diagonals of the parallelogram. Thus, the line F_KT_K tangent to the ellipse is parallel to the diagonal A_KC_K. This is due to the fact that F_KT_K and A_KC_K are projections of two parallel straight lines FT and AC.

The graphic constructions that precede drawing the ellipse are shown in Fig. 319. It is advisable to perform them in the following sequence:

1. Construct the axonometric projection of the square (the parallelogram $A_KB_KC_KD_K$) and draw the diagonals A_KC_K and B_KD_K.

2. Note the midpoints of the sides of the parallelogram: points 1_K, 2_K, 3_K and 4_K.

3. Construct a right isosceles triangle 1_KLB_K on line-segment 1_K-B_K as hypotenuse.

4. From point 1_K with radius 1_K-L describe a semicircle such that will intersect A_KB_K in points M and N. These points divide

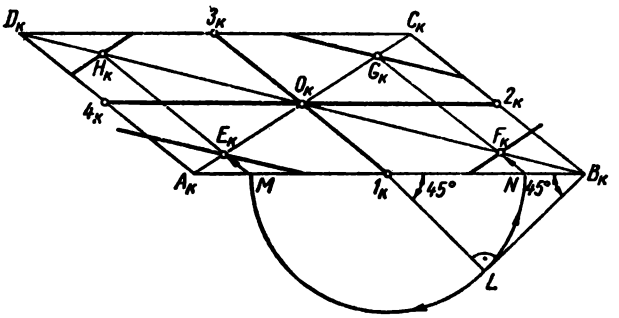


Fig. 319

the line-segments A_K-1_K and 1_K-B_K (they are equal) in the ratio 7 : 10.

5. Through points M and N draw straight lines parallel to the lateral sides of the parallelogram and note points E_K, F_K, G_K and H_K on the diagonals.

6. Construct lines tangent to the ellipse at the points found. Two tangent lines will be parallel to the diagonal A_KC_K, the other two must be parallel to B_KD_K.

With 8 points and the same number of tangent lines it is possible to draw an ellipse with sufficient accuracy.

In the general case, determining the axes of the ellipse is associated with regularities of perspective-affine correspondence, which are discussed in the chapter "Geometric Transformations". There we shall also consider the construction of the axes of the ellipse.

And now let us examine some of the properties of a rectangular axonometric projection of a circle. We shall show that the major axis of the ellipse, which is the rectangular projection of the circle, is perpendicular to the projection, on the same plane, of the normal to the plane of the circle. Let there be given a circle of diameter d

with centre O (Fig. 320), the plane P of which forms with the projection plane K an angle φ, and the normal ON to the P plane.

The orthogonal projection of a circle is an ellipse. Its major axis A_KB_K is a projection of the same diameter AB of the circle that is parallel to plane K: i. e., $A_KB_K=d$. All other diameters are projected with distortion, they are foreshortened.

Consider the angle $N_KO_KB_K$. It is the projection of the right angle NOB, one of the sides of which is parallel to the K plane (OB is

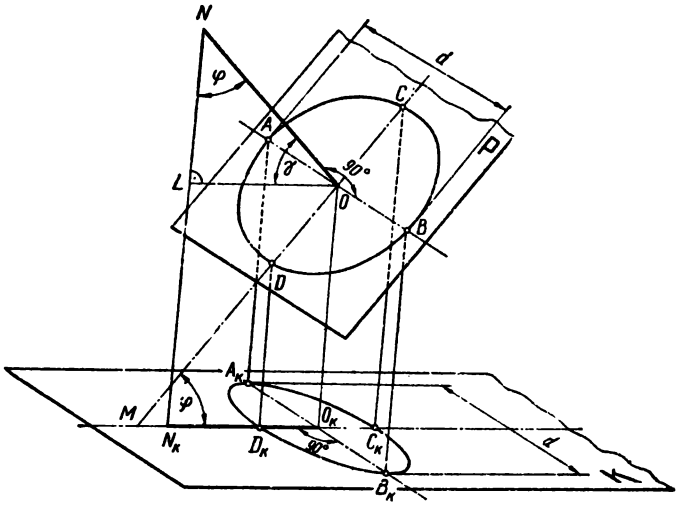


Fig. 320

parallel to K). Hence, O_KB_K is perpendicular to O_KN_K, that is, the major axis of the ellipse is perpendicular to the projection of the normal ON.

It may readily be shown that the minor axis C_KD_K of the ellipse is parallel to the projection of the normal to the plane of the circle. Indeed, C_KD_K and N_KO_K are perpendicular to A_KB_K, and therefore C_KD_K is parallel to N_KO_K. The length of the minor axis C_KD_K will be $d\cos\varphi$.

To summarize, then: $2a=d$, $2b=d\cos\varphi$ where $2a$ is the major axis of the ellipse and $2b$ is the minor axis.

In making axonometric drawings, it is frequently necessary to construct circles located in the coordinate planes. Then the absent axis will be the normal to the plane of the circle. For instance, if the circle lies in the xy-plane, the z-axis will be the normal to this plane. On the basis of the property proved above, the major axis

of the ellipse (which is the orthogonal projection of the circle) must be perpendicular to $O_K z_K$.

Hence, in the case under consideration, the major axis of the ellipse is perpendicular to the coordinate axis that does not lie in the plane of the circle.

We again stress that under consideration is the construction of a rectangular axonometric projection of a circle located in one of the coordinate planes or in planes parallel to the coordinate planes.

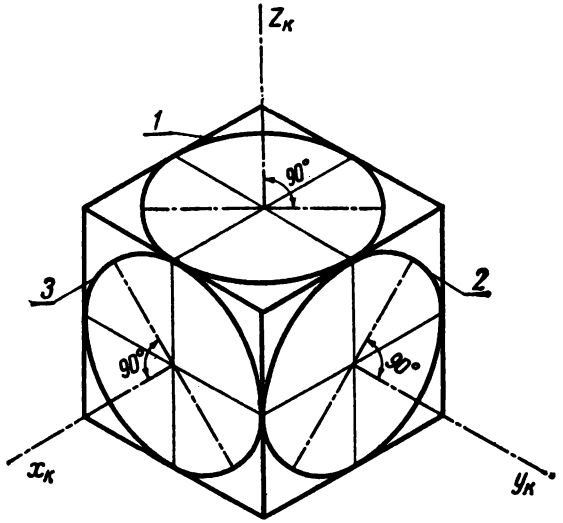


Fig. 321

Figs. 321 and 322 illustrate ellipses that are rectangular isometric and dimetric projections of circles inscribed in the faces of two cubes. The faces of these cubes are parallel to the corresponding coordinate planes.

The major axis of ellipse *1* is perpendicular to the axonometric projection z_K of the z-axis, which means the major axis is horizontal. The major axes of ellipses *2* and *3* are perpendicular to y_K and x_K, respectively. It remains to determine the dimensions of the axes. Formula (2) holds only for the case when the distortion factors satisfy the relation $s^2+t^2+u^2=2$.

Yet, in accord with the Soviet Standard, we construct the rectangular isometric projection without foreshortening along the coordinate axes; and this corresponds to an enlargement of the image by a factor of 1.22. Which means that the major axes of the ellipse

in Fig. 321 must also be increased over the diameter of the circles the same number of times: $2a=1.22d$.

It will readily be seen that the minor axis $2b=0.7d$. Indeed, suppose one of the coordinate planes, say xOy, coincides with plane P (see Fig. 320). Then the normal ON will be parallel to the axis that is absent from plane P; thus, ON is parallel to Oz.

The distortion factor u of this axis is determined by the ratio $\frac{O_KN_K}{ON}=\cos\gamma$.

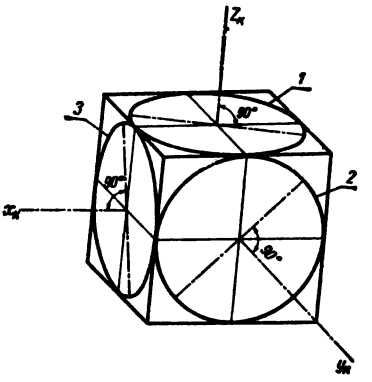


Fig. 322

From the right triangle NLO it follows that $\gamma+\varphi=90°$, and therefore $\cos\varphi=\sqrt{1-\cos^2\gamma}$.

Since in rectangular isometry $s=t=u=\sqrt{\frac{2}{3}}$, $\cos\gamma=\sqrt{\frac{2}{3}}$ ($\cos\gamma=u$). Substituting this number into formula $2b=d\cos\varphi$ yields $2b=\frac{\sqrt{3}}{3}d$. Noting the above-mentioned increase in dimensions of the image by a factor of 1.22, we get $2b\cong 0.7d$.

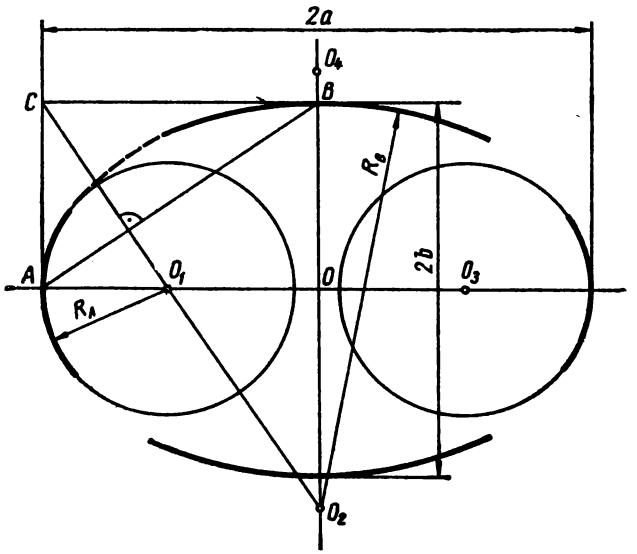


Fig. 323

Performing analogous computations for rectangular dimetry (see Fig. 322), we get the following results. The major axis of each of the ellipses *1*, *2* and *3* must be taken equal to 1.06 of the diameter of the circle. The ratio of the lengths of the axes of ellipses *1* and *3* is 1 : 3; of ellipse *2*, 9 : 10.

When drawing an ellipse from given axes it is advisable to utilize the centres of curvature of its vertices: points O_1, O_2, O_3 and O_4 (Fig. 323). To do this, draw AC parallel to OB, BC parallel to OA and from point C (the point of intersection of the lines AC and BC) drop a perpendicular onto AB.

Extending the perpendicular to its intersection with the axes of the ellipse, determine the centres of curvature O_1 and O_2 of vertices A and B. Taking advantage of the property of central symmetry of an ellipse, find points O_3 and O_4. The arcs of circles of radii R_A and R_B may be joined by using a French curve as shown in Fig. 323 by the dashed line. This method of approximate construction of an ellipse ensures the symmetry of the image relative to the axes with only a slight deviation from the actual shape.

Sec. 63. The Construction of Axonometric Views

It is suggested that the transition from orthogonal projections of an object to axonometric representation be done in the following sequence:

1. On the orthogonal drawing, set up the axes of a rectangular coordinate system; refer the given object to these axes. Orient the coordinate system so as to facilitate measuring the coordinates of points of the object. For example, when constructing the axonometry of a solid of revolution, it is advisable to have the axis of the solid coincide with one of the coordinate axes.

2. Plot the axonometric axes so as to ensure the best possible pictorialness of representation and visibility of appropriate points of the object.

Practical experience in the construction of axonometric projections indicates that the best pictorial views are obtained in rectangular trimetry with a ratio of distortion factors $s : t : u = 0.9 : 0.5 : 1.0$. For a comparatively small width of the object, the distortion factor along the axis $O_K y_K$ must be increased.

The best results in pictorialness are obtained by rectangular dimetry using a 1 : 0.5 : 1 ratio of distortion factors.

Of oblique axonometric projections, wide use is made of vertical (K parallel to V) and military perspective (K parallel to H) because of their simple construction. As has already been noted, vertical projections do not distort the shape of the facade, while military perspective retains unchanged all contours of the plan.

3. Draw the secondary projection on one of the orthogonal projections of the object.

On what coordinate plane is it advisable to construct the secondary projection? The answer should proceed solely from the relative simplicity of the graphical constructions. It is advisable to draw that secondary projection of the object which is the simplest.

If shadows have to be shown on an axonometric drawing, the secondary projection is constructed on the horizontal plane $x_K O_K y_K$.

4. Construct the axonometric representation.

The scales of the orthogonal drawing and axonometric drawing may differ. Thus, in the construction of rectangular dimetry of an

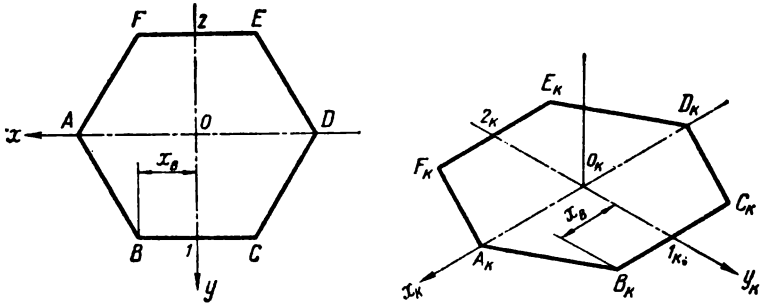


Fig. 324

object ($s : t : u = 1 : 0.5 : 1$) all linear dimensions along the axis $O_K x_K$ and $O_K z_K$ may be increased or decreased n times, but then the dimensions along the axis $O_K y_K$ must be accordingly increased or reduced by a factor of $0.5n$.

Thus, it is advisable to begin an axonometric representation of an object with the secondary projection; that is, with the construction of the axonometry of a plane figure, which is a front or top view of the given object. For this reason, let us first consider the construction of axonometry of figures located in the planes of projection.

Example 1. *To construct a rectangular isometric projection of a regular hexagon ABCDEF* (Fig. 324).

Here, for the coordinate axes it is best to take the axes of symmetry of the hexagon (the x- and y-axis). To construct the isometric projection, lay off line-segments $O_K A_K = O_K D_K = OA$ from the origin of axonometric axes (point O_K) along the axis $O_K x_K$ (the distortion factors along all axes are taken to be unity).

Noting that the remaining vertices are symmetrical in pairs about the coordinate axes, lay off segments O_K-1_K and O_K-2_K from

point O_K along the axis $O_K y_K$; they are equal to one another and to the segments O-1 and O-2. Through points 1_K and 2_K draw straight lines parallel to the axis $O_K x_K$; here determine the positions of points B_K, C_K, E_K and F_K. This is done by means of the equal segments 1_K-B_K, 1_K-C_K, 2_K-E_K, 2_K-F_K whose lengths correspond to the x-coordinate of the vertices B, C, E and F of the hexagon.

Example 2. *To construct a rectangular dimetric projection of curve AB located in the xz-plane* (Fig. 325).

The axonometric projection of a plane curve is constructed from the coordinates of separate points. To illustrate, take a curve AB

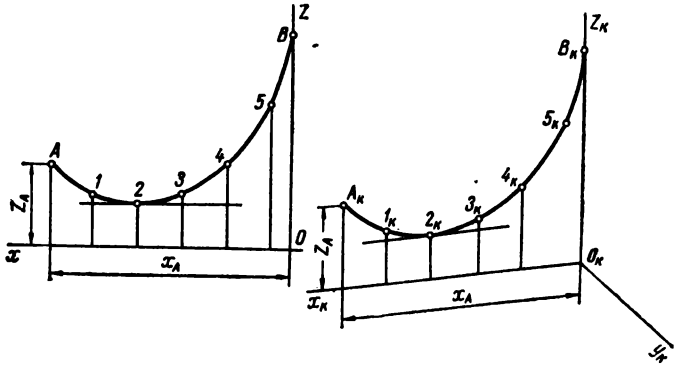


Fig. 325

with indicated points 1, 2, 3, etc., and defined rectangular coordinates. Passing to the axonometric drawing, take into account the distortion factors. In our case, $s=u=1$, and therefore the axonometric coordinates of each point of the curve are equal to the rectangular coordinates.

Let us now begin the construction of the axonometry of solids

Example 3. *To construct a rectangular isometric projection of a truncated hexagonal prism with a through rectangular aperture* (Fig. 326).

First, on an orthogonal drawing set up a system of coordinate axes $Oxyz$ such that the origin is in the centre of the prism base.

We dispense with a description of the construction of the secondary projection of the prism because it repeats exactly what was given in Example 1 in the construction of a regular hexagon.

It will be sufficient to note that the through rectangular aperture requires that we show on the secondary projection two lines of the hidden contour $a_K b_K$ and $c_K d_K$. The next stage in the work (constructing an axonometric view of the prism) consists in the following:

through points 1_K, 2_K, ..., 6_K, a_K, b_K, c_K, d_K of the secondary projection draw vertical straight lines; on the latter lay off segments equal to z_1, z_2, ..., z_6, z_A and z_E.

Join with straight lines the points obtained (the upper extremities of the vertical segments) in such a manner as to ensure correspondence between the orthogonal and axonometric representations of the prism.

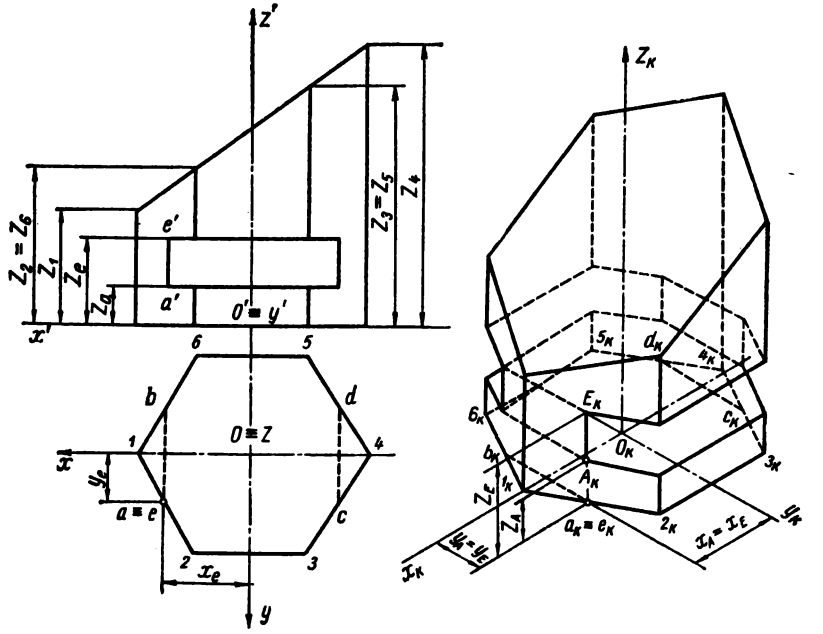


Fig. 326

In Fig. 326 this one-to-one correspondence is illustrated by points A and E and the line-segment that connects them.

Example 4. *To construct a rectangular dimetric projection of a reinforced concrete assembly* (Fig. 327).

In the orthogonal drawing, arrange the coordinate axes so that the assembly lies in one of the lower octants, say IV. This orientation will enable us to show how the girders mate with the column and plate. In the construction of the secondary projection and axonometry of the assembly, the linear dimensions along the axes $O_K x_K$ and $O_K z_K$ were doubled, but remained unchanged along the axis $O_K y_K$. This complied with the condition which, in the case of rectangular dimetry, must be satisfied by the ratio of distortion factors $s : t : u = 1 : 0.5 : 1$.

Using point E we can trace the transition from the orthogonal drawing to the axonometric representation.

The axonometric view of the assembly in Fig. 327 is displaced downwards somewhat and detached from the secondary projection for the sole purpose of retaining in the drawing the successive stages in the construction.

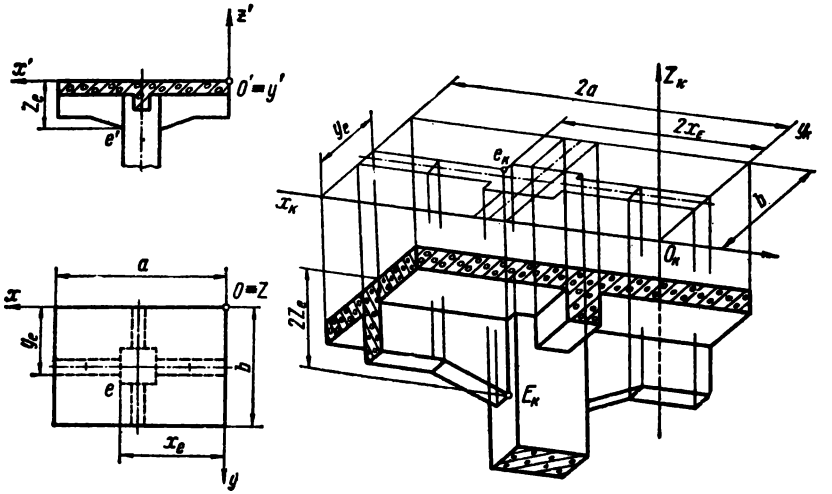


Fig. 327

Example 5. *To construct a rectangular dimetric projection of a cylinder with a through triangular aperture* (Fig. 328).

Place the coordinate origin in the centre of the lower base, and the z-axis on the cylinder axis. Here, it is advisable to construct the secondary projection on the coordinate plane $x_K O_K z_K$, since this will necessitate the construction of straight lines only. Parallelogram $A_K B_K B_{1K} A_{1K}$ and triangle $C_K D_K E_K$ serve as such a projection. To enhance the pictorialness of the drawing, the cylinder is depicted with a quarter cut out.

We will not dwell on drawing the ellipses (the contours of the upper and lower bases of the cylinder) and will consider the construction of one of the points N_K of the elliptical arc along which the inclined plane of the triangular aperture intersects the cylindrical surface.

We can also find the axonometric projection N_K of point N both with the aid of the orthogonal drawing and independently.

In the former case, on the orthogonal projections determine two rectangular coordinates x_N and y_N of this point. Then using the

coordinate x_N on the straight line $D_K E_K$, show the secondary projection n'_K. Finally, through n'_K draw a straight line parallel to the axis $O_K y_K$ and lay off on it from point n'_K a segment equal to $0.5\,y_N$ (in our case, $t=1:2$). In other words N_K is constructed by the coordinate method.

This point may also be determined by the intersection of the generatrix $L_K L_{1K}$ of the cylinder and the horizontal line $n'_K N_K$ of the inclined plane of the aperture; both generatrix and horizontal

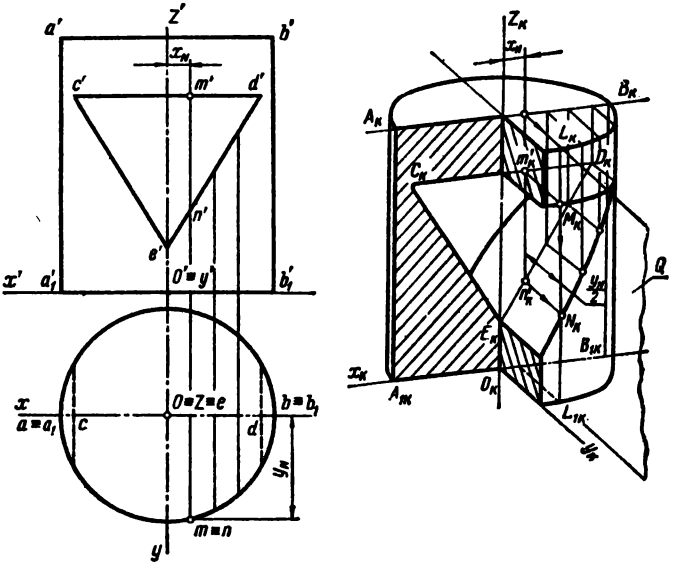


Fig. 328

line must lie in a single plane, Q, which is parallel to the coordinate plane $y_K O_K z_K$. In Fig. 328 we similarly find the other points of the curve of the oblique section.

Example 6. *The rectangular isometric projection of a sphere* (Fig. 329).

The projectors used to generate the axonometric projection of a sphere form the surface of a circular cylinder.

The picture plane may cut this cylinder either in a circle or in an ellipse. The latter occurs in an oblique projection direction.

In our case, the picture plane is perpendicular to the generatrices of the projecting surface, and the contour of the projection of the sphere is a circle. Its radius is equal to that of the sphere.

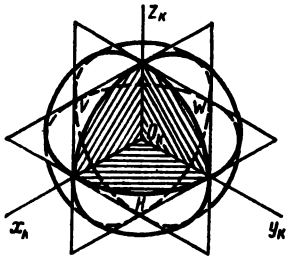


Fig. 329

To give the object in Fig. 329 volume, we show sections of the sphere by coordinate planes, and the 1/8th of the sphere located in the first octant is assumed to be at a distance.

To construct the sections (three equal ellipses), lay off on the coordinate axes line-segments equal to $0.82R$ (R is the radius of the sphere). Enlarging these segments to R would have required increasing the radius of the circle (which is the contour of the axonometric projection of the sphere) the same number of times, i. e., by a factor of 1.22.

Example 7. *To construct an oblique dimetric projection of half of a cam clutch with a quarter cut out* (Fig. 330).

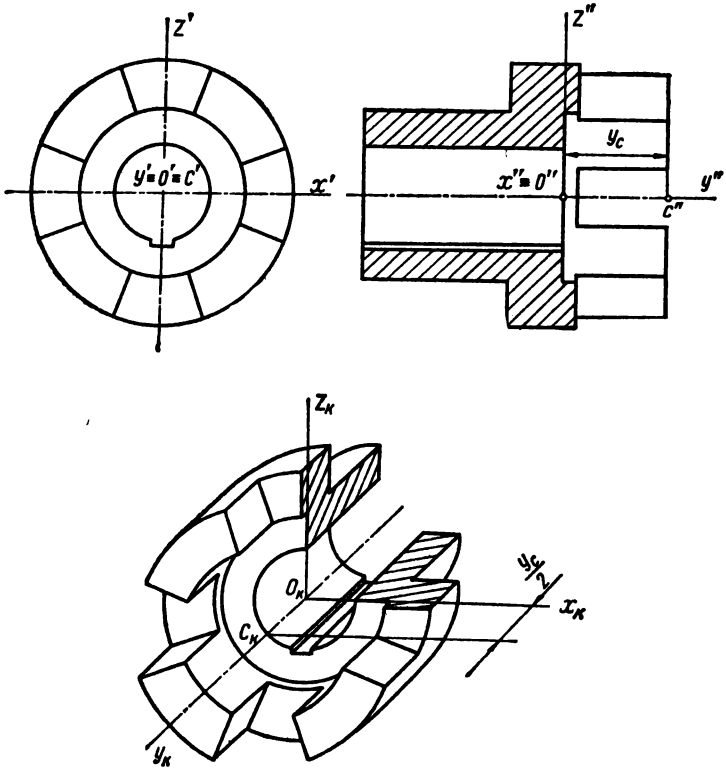


Fig. 330

Since in this case the plane of the axonometric projections is parallel to the vertical plane V, all faces of the object parallel to V will be depicted true size in axonometric. It is best to place the origin in one of the points of the axis of the half-clutch. Let it be point O located in the plane from which the key groove begins. The centres of the remaining circles will be displaced along the axis $O_K y_K$ away from the origin. The displacement of each centre will be determined by its y-coordinate reduced by one-half (the distortion factor along the axis $O_K y_K$ is 0.5). Now, to construct the outer contour of the end faces of the cams, it is necessary, on the axis $O_K y_K$, to take a point C_K at a distance $\frac{y_C}{2}$ from the coordinate origin. In a similar manner we find the centres of the other circles.

To make the representation of the half-clutch more pictorial, a section is made by two planes, thus illustrating its inner structure.

Note that it is advisable to start the construction of the axonometry of an object having a 1/4 cutaway with the production of figures (sections) that are located in the cutting planes. We illustrate this in the following example.

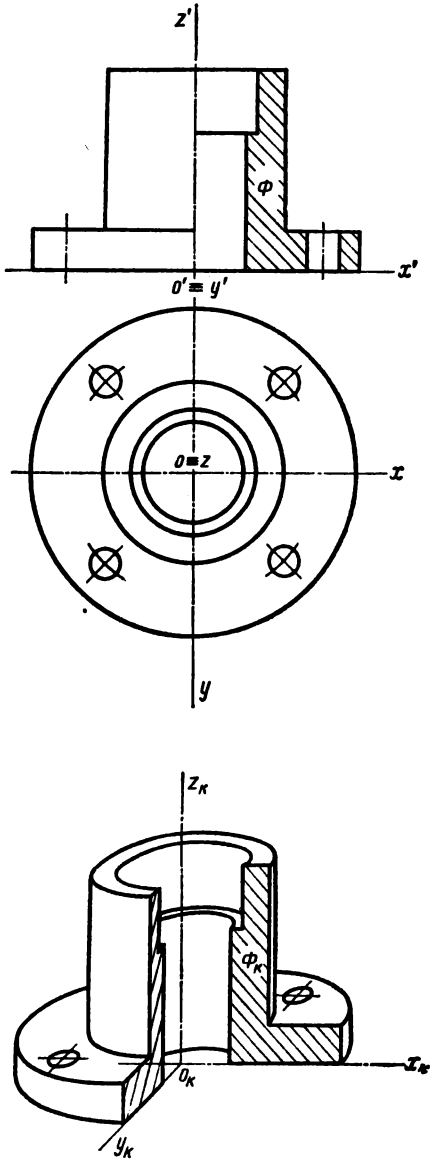


Fig. 331

Example 8. *To construct an oblique dimetric projection of the object* depicted in Fig. 331.

First draw the section of the object located in the coordinate plane $x_K O_K z_K$. Here, when $s=u=1$, the figure does not differ at all from the cross-hatched figure Φ in the vertical projection. Then construct the section in the plane $y_K O_K z_K$. All altitude dimensions of this section are transferred from the orthogonal drawing, and the dimensions along the axis $O_K y_K$ are foreshortened by one-half. It now remains to construct the lines of the visible contour of the article. In our case they are circles, the axonometric representation of which was given in Sec. 62.

Example 9. *To construct oblique dimetric projections of elements of the wooden assembly* shown in Fig. 332.

The oblique pieces in one plane adjoin the support and in the other rest on the end of a cushion.

Let the coordinate origin be in the point of intersection of the axes of the oblique pieces, and let the z-axis coincide with the axis of the support. Since the length of the support is not given in the orthogonal drawing the centre of the upper end may be located on the z-axis so that its z-coordinate is greater than z_A.

We omit any description of the construction of the end contour (see Sec. 62) and note that the points of the lines of intersection of the cylindrical support by three planes are constructed from the coordinates of these points. For instance, the upper point A_K on the cylinder generatrix located in plane $x_K O_K z_K$ is determined with the aid of segment z_A. The axonometric representation B_K of point B is constructed with account taken of the distortion factors along the coordinates x_B, y_B and z_B. Having determined B_K, it is easy to find point C_K, which is symmetric to B_K about the plane $x_K O_K z_K$.

The axonometric projection of the cushion is produced in a manner similar to the construction of a rectangular isometric hexagonal prism (see Example 3 in Sec. 63). In this case, however, we first construct the upper base. It will be noted that the plane of the inclined cut of the cushion appears as a straight line in the axonometric drawing.

We examine the right angle brace of the assembly. The coordinate plane $x_K O_K z_K$ is the plane of symmetry of this element. It is advisable here to construct the secondary projection of the angle brace: the figure $1'_K 2'_K 3'_K 4'_K 5'_K 6'_K$. The plane $x_K O_K z_K$ will also contain the axis of the oblique piece which coincides with its secondary projection $e'_K f'_K$. Then, through points $2'_K$ and $3'_K$ draw straight lines parallel to the axis $O_K y_K$; on them lay off line-segments equal to $\frac{1}{2}y_2$ and $\frac{1}{2}y_3$ in both directions from the indicated points.

The coordinates y_2 and y_3 are taken from the orthogonal drawing by using the axis view of the oblique piece. In exactly the same way we can construct any pair of points of elliptical arcs located on a

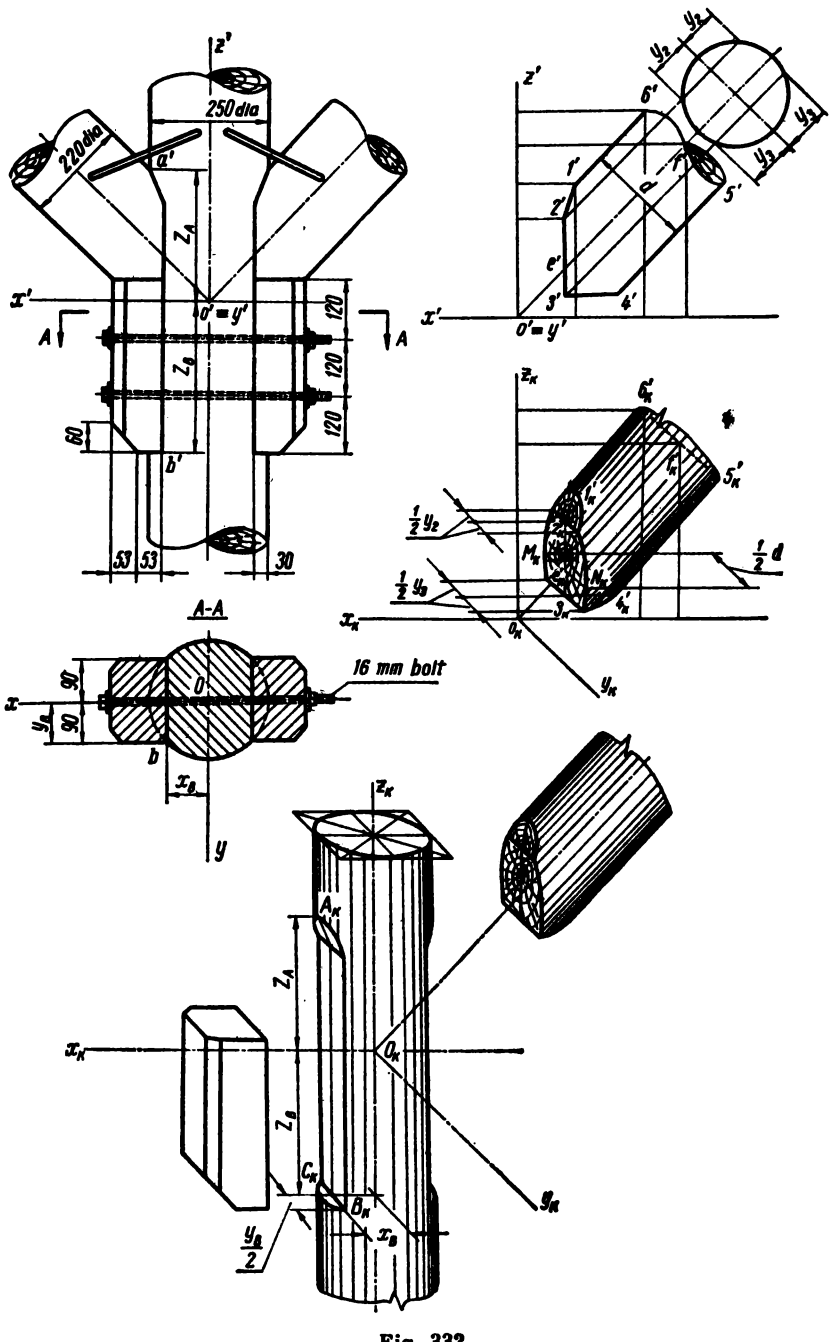

z'
250 dia
220 dia
a'
z_A
x'
A
o' = y'
120
120
120
z_B
60
b'
53
53
30
A-A
16 mm bolt
90
90
x
y_A
o
b
x_B
y
z_K
A_K
z_A
x_K
O_K
z_B
C_K
B_K
y_K
$\frac{y_B}{2}$
x_B
y_2
6'
f'
e'
d'
5'
3'
4'
$\frac{1}{2}y_2$
$\frac{1}{2}y_3$
$\frac{1}{2}d$
M_K
N_K
O_K

Fig. 332

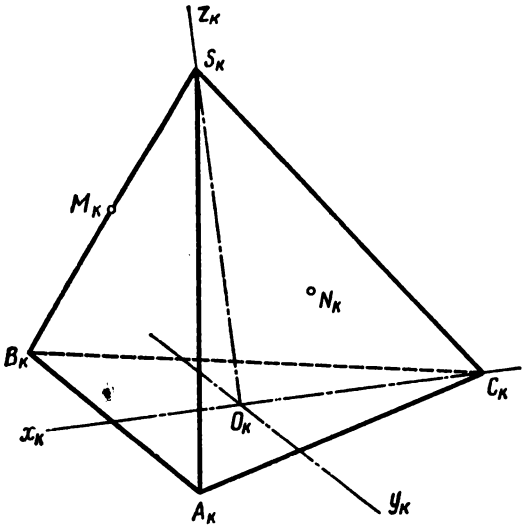


Fig. 333

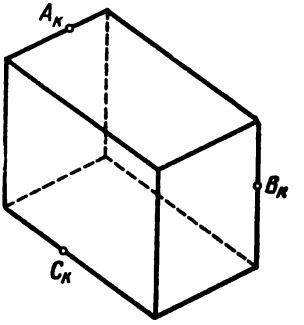


Fig. 334

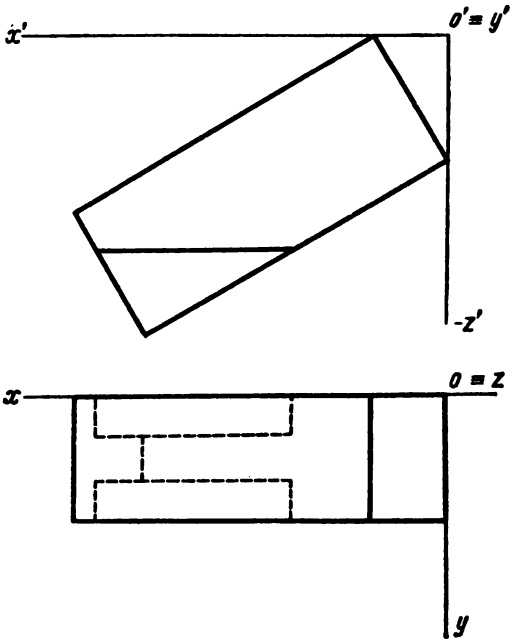


Fig. 335

single chord parallel to the axis $O_K y_K$ (see, for instance, points M_K and N_K which belong to the chord that intersects the axis of the oblique piece in point E). We draw the outline generatrices of the

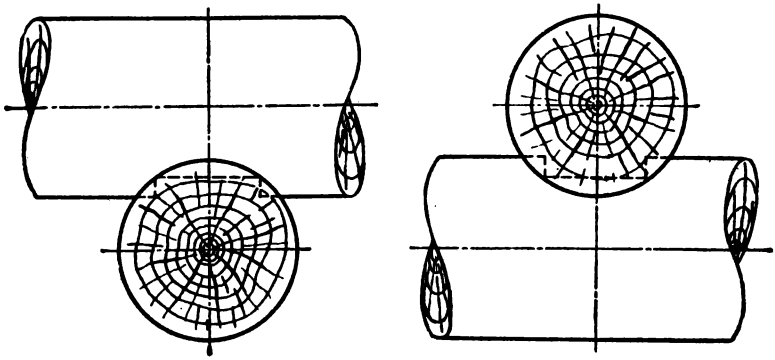


Fig. 336

axonometric projection of the cylindrical oblique piece as tangent lines to the elliptical sections parallel to the axis $e'_K f'_K$.

In Fig. 332, the cushion and oblique piece are shown in a displaced position. The former is displaced in the direction of the axis $O_K x_K$, and the latter along its own axis. This displacement brings out better the shape of each element.

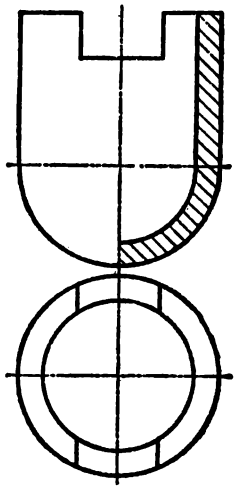


Fig. 337

Exercises

1. To determine the coordinates of points M_K and N_K, M_K being located on edge $S_K B_K$ and N_K on face $S_K A_K C_K$ of a tetrahedron (Fig. 333).
2. Given an axonometric view of a rectangular parallelepiped. Required to construct its section by a plane specified by three points A_K, B_K and C_K (Fig. 334).
3. To construct an axonometric representation (vertical oblique isometry) of an inclined block specified by orthogonal projections (Fig. 335).
4. To construct a rectangular isometric projection of the elements of an assembly (Fig. 336) while ensuring optimal visibility of mating.
5. From given orthogonal projections of an article (Fig. 337), to construct its rectangular isometry with a one-quarter cutaway.

CHAPTER XIII

LINEAR PERSPECTIVE

Sec. 64. Essentials of the Method

Perspective* is a method of representing solids and plane figures based on the employment of *central projection* (see Secs. 1 and 2). To construct the perspective of an object from some point S (point of sight) we draw projectors to all points of the object. In the path

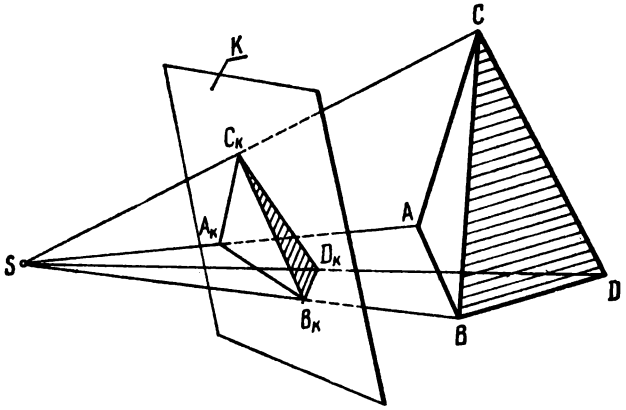


Fig. 338

of these projectors we place a surface K (the picture plane) on which we construct the desired image by determining the points of intersection of the rays with the picture plane. Fig. 338 illustrates the principle of constructing the perspective of an object on a plane (linear perspective).

Representation of objects by means of central projection is very pictorial. To a certain approximation, the perspective of an object corresponds to what the human eye perceives; that is to say, it con-

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*Perspicere (perspectus, *pp.* of perspicere) in Latin means "to look through". Perspective is also the representation of objects constructed with apparent changes in dimensions and contours caused by the distance of the objects from viewing point and their position in space.

veys the apparent changes in the size and shape of the object caused by its location and the distance from the viewer. This is due to the fact that the process of vision is geometrically identical with the method of central projection.

The surface on which a perspective representation is made need not be plane, it may be cylindrical or spherical. For a cylindrical surface, the perspective is panoramic, for a spherical surface, spherical. We shall deal only with linear perspective.

In Sec. 2 we discussed central projection and found that a single central projection of a point (and, hence, an object) does not determine its position in space. Indeed, to each point A_K on a plane K

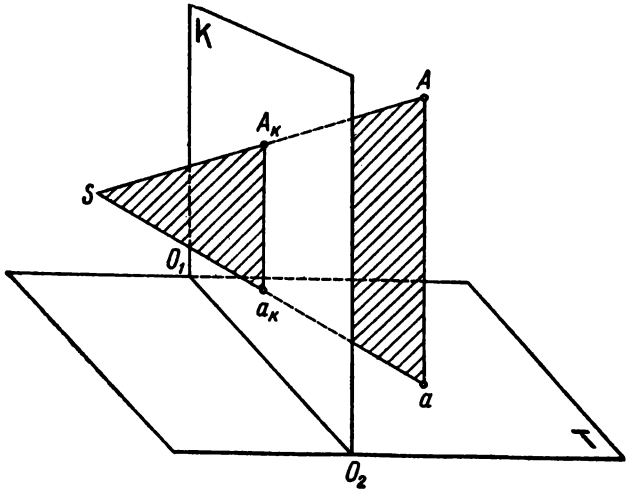

Fig. 339

(see Fig. 339) there will correspond any point of the projecting ray SA_K.

To ensure one-to-one correspondence between the points of an object and the points on the picture plane K (to make the representation reversible), we do as follows. We project the given point A (Fig. 339) orthogonally on a horizontal plane T perpendicular to plane K, and then on the picture plane we determine the perspective (central) projections both of point A and of its horizontal projection a. In Fig. 339, a ray directed at A intersects the picture plane in point A_K. This is the perspective projection (or simply perspective) of point A. The second ray moving to a intersects the picture plane in point a_K and determines the perspective of the horizontal projection of A. Let us agree to call point a_K the secondary projection of point A

(the primary projection being a). In the K plane, the perspective of a point and its secondary projection lie on a single vertical line. This is because the straight line $A_K a_K$ is the line of intersection of two vertical planes: the picture plane and the radial plane (in Fig. 339 the latter is cross-hatched).

The radial plane is vertical in space because it passes through the perpendicular Aa to plane T. We now show that the perspective of a point and its secondary projection uniquely determine the position of a point in space. In Fig. 339 let there be given points A_K, a_K, S and two planes K and T. Draw rays from S to A_K and a_K. The intersection of the latter (Sa_K) with plane T will yield the primary projection a. Erecting a perpendicular to T at the point obtained (a), we get the intersection of the perpendicular with ray SA_K. This will be the desired point A.

Sec. 65. The System of Planes in Linear Perspective

In this section we shall discuss the apparatus of linear perspective.

First let us consider four planes K, T, N_0 and H (Fig. 340). K is the vertical plane of the picture; on it we construct the perspective representation of objects by locating them ordinarily in the horizontal plane T. T is therefore called the ground plane. Planes K and T are intersected at right angles along line O_1-O_2, which is called the ground line. The two other planes, H and N_0, pass through the point of sight S parallel to T and K respectively. The horizontal plane H is called the horizon plane, and the vertical plane N_0, which is parallel to the picture plane, is called the neutral plane. Planes H and K intersect along the horizontal line h-h. This is the horizon.

Drop from S a perpendicular onto the ground plane. The foot of this perpendicular is known as the station point. The distance from S to the picture is called the principal distance, and point P—the foot of the perpendicular from the point of sight S to plane K—is the principal point of the picture. Since a perpendicular from S on the picture plane belongs to plane H, the foot of this perpendicular (principal point) will lie on the line of the horizon. In certain cases we have to employ so-called points of distance D located on the horizon so that $DP=SP$. The space bounded by planes K and N_0 is called the intermediate space, the part of space in front of the viewer and behind the picture, the object space, and the space behind the viewer, imaginary space.

To construct perspective we must know the position of S relative to planes K and T.

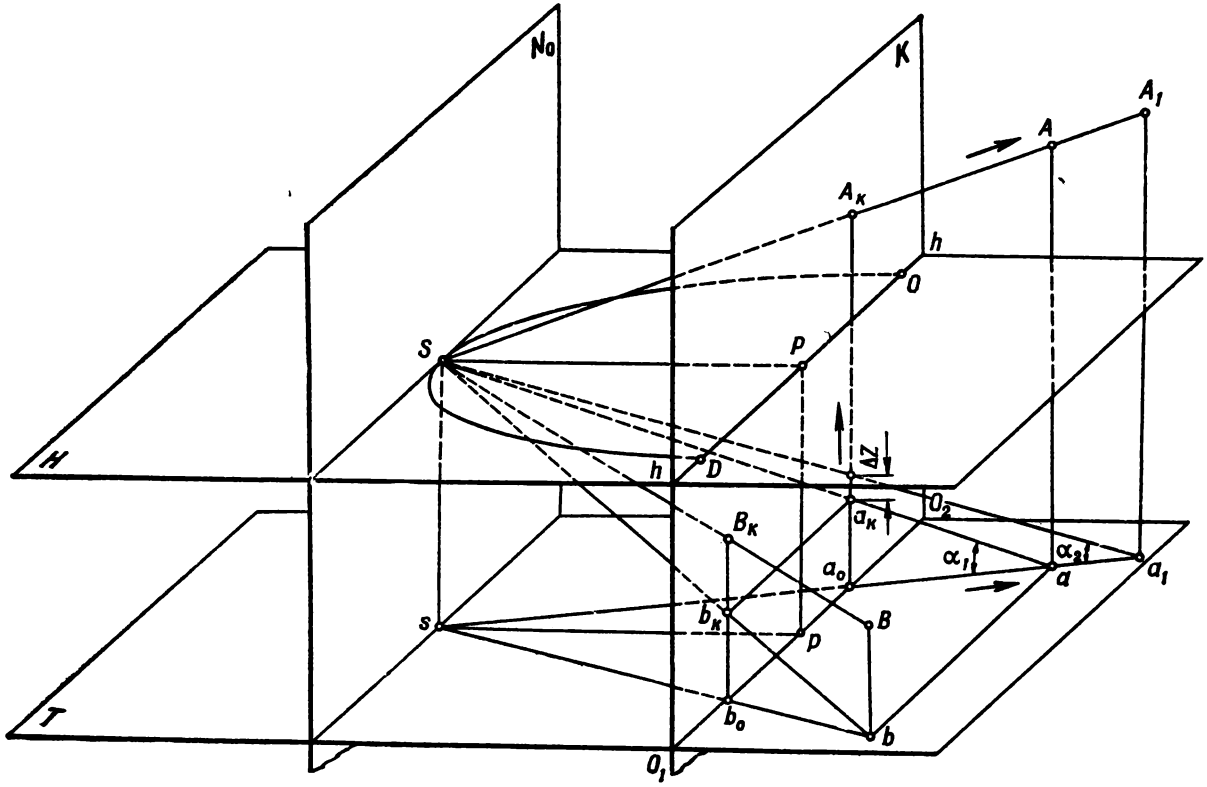

Fig. 340

Sec. 66. The Perspective of a Point Located in Different Parts of Space

Let us consider point A in object space (see Fig. 340) and see what changes its perspective and secondary projection undergo during the motion of A along the projecting ray SA. Let the point move from A to A_1. Its perspective will remain in point A_K. The secondary projection will be displaced vertically upwards by Δz. As point A recedes from the picture plane, its secondary projection will approach the horizon, since the angle of inclination of the ray that determines the secondary projection diminishes in this motion ($\alpha_2 < \alpha_1$). In the limit, when point A recedes to infinity, angle α will be equal to zero and a ray directed to the horizontal projection of an infinitely distant point will lie in the H plane.

Consequently, the secondary projection of an infinite point of object space must lie on the horizon. This very important conclusion will frequently be utilized in what follows.

Comparing the positions of points A and A_1 relative to the picture plane with their secondary projections, we conclude that the closer a point is to the picture, the smaller is the distance between its secondary projection and the ground line. If two points, A and B, are at equal distances from the picture plane, their secondary projections lie at one and the same distance from the ground line.

Indeed, in this case line ab is parallel to the ground line and in triangle abs we have two equal ratios: $\frac{aa_0}{as} = \frac{bb_0}{bs}$.

But since $\frac{aa_0}{as} = \frac{a_0a_K}{sS}$ (from triangle asS) and $\frac{bb_0}{bs} = \frac{b_0b_K}{sS}$ (from triangle bsS), it follows that

$$\frac{a_0a_K}{sS} = \frac{b_0b_K}{sS}$$

whence we get $a_0a_K = b_0b_K$.

Fig. 341 shows the construction of perspectives and secondary projections of points located in intermediate (point B) and imaginary (point E) space. The secondary projections of the points of intermediate space are below the ground line (b_K, for instance). But if the points belong to imaginary space, their secondary projections will lie above the horizon (for example, e_K). The secondary projections of points of object space can lie only between the ground line and the horizon (see Fig. 340). Finally, at the ground line will be the secondary projections of points lying on the picture plane (for example, point N in Fig. 341). The horizon is the locus of secondary projections of infinitely distant points of space. Thus,

the secondary projection of a point gives a clue to the space in which the given point lies.*

We shall agree to designate points of space by capital letters (A, B, E, ...), their perspectives by the same letters with a subscript (A_K, B_K, E_K, ...), and their secondary projections, by the corresponding lower-case letters (a_K, b_K, e_K, ...).

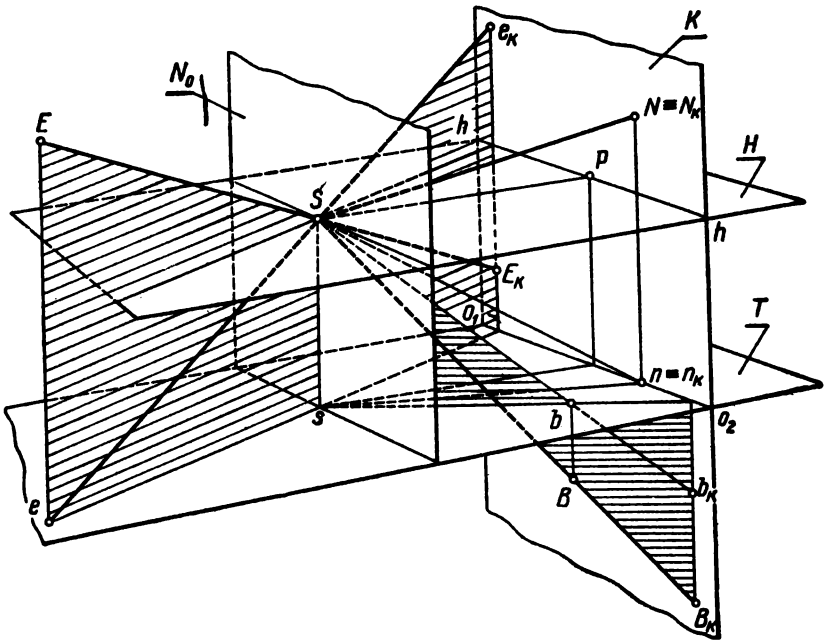

Fig. 341

Sec. 67. The Perspective of a Straight Line

The projecting rays that pass through point S and some straight line AB form a plane. This radial plane intersects the picture along a straight line A_KB_K, which is the perspective of the given line (Fig. 342). When the line passes through the point of sight S, its perspective degenerates into a point. Specification of the perspective of a straight line does not alone determine the line's position in space. A perspective representation of a straight line will be reversible if it is supplemented by a secondary projection.

*With the exception of points at infinity.

In Fig. 342 the perspective of line AB and its secondary projection are defined by perspectives and the secondary projections of two of its points, A and B.

Having A_KB_K and a_Kb_K, it is possible to determine two characteristic points of the line: the perspective F_K of an infinite point F

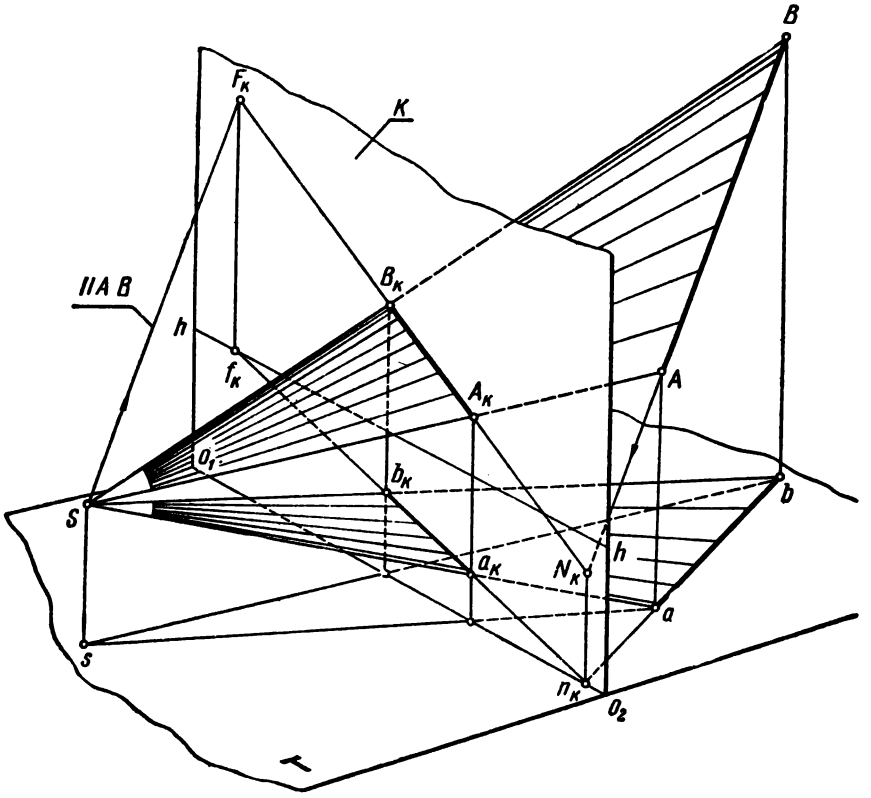

Fig. 342

and the beginning of line N_K (the beginning of a straight line is usually the point of intersection of the line with the picture). The secondary projection of the former (point f_K) must lie on the horizon, while that of the latter must lie on the ground line (point n_K). Drawing a vertical straight line through f_K to its intersection with A_KB_K, we get the perspective F_K of an infinite point of the line. A projecting ray directed towards an infinite point of the given straight line AB (parallel to AB) will intersect the picture at this point. A perpend-

icular to the ground line O_1-O_2 passing through n_K will intersect A_KB_K and determine the beginning of the line (point N_K).

The beginning and the infinite point of a straight line are ordinarily utilized in constructing the perspective of a variety of objects.

The position of perspective of an infinite point of a line on a picture enables us to judge the position of the straight line in space. Thus, if point F_K lies above the horizon (see Fig. 342), then the line AB is an ascending line, since a ray directed from the point of sight parallel to the given straight line moves upwards. If point F_K lies below the horizon, then line AB is a descending line (Fig. 343). Point M_K, in which the perspective of the straight line intersects the secondary projection, is the perspective of the trace of AB on the ground plane T. Finally, if the perspective of an infinite point of the line lies on the horizon, then line AB is in a horizontal position (Fig. 344).

We will frequently have to construct straight lines that are perpendicular to the picture plane. To find the infinite point of such lines, draw a ray from the point of sight perpendicular to the picture plane. A ray of this kind will intersect the picture in the principal point. Therefore, the principal point P is the perspective of infinitely distant points of straight lines perpendicular to the picture (see line N_KP in Fig. 344).

Fig. 345 shows straight lines, all the points of which are an equal distance from the picture plane. As was demonstrated in Sec. 66, the secondary projections of such points must be the same distance from the ground line. That is why the secondary projections of line AB, parallel to the picture, and line CL, which is parallel to the picture and the ground plane, are drawn parallel to the ground line O_1O_2.

When line EF is perpendicular to the ground plane, its secondary projection becomes a point (see Fig. 345). But if the line lies in the ground plane, the perspective of the line and its secondary projection coincide.

Sec. 68. The Relative Positions of Two Straight Lines

1. *Parallel lines.* If lines in space are parallel, their perspectives pass through a common vanishing point. Indeed, let us consider the construction of the perspective of the parallel lines AB and LE shown in Fig. 346. Extending each one to its intersection with the picture, we find their beginnings: points N_{1K} and N_{2K}. The second point that determines the desired perspectives is the common infinite point F, for the construction of which we draw a ray from the point of sight S parallel to the given lines.

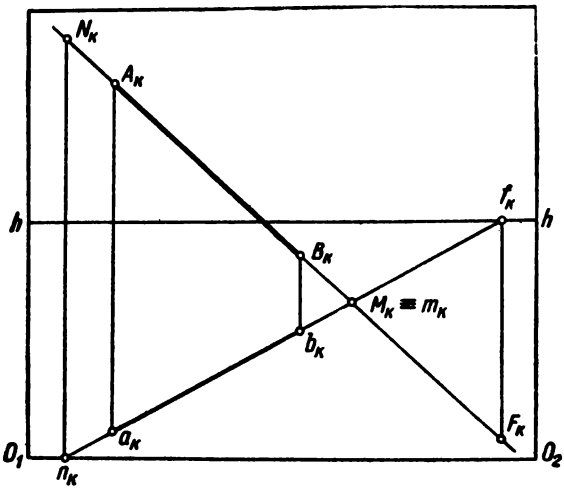

Fig. 343

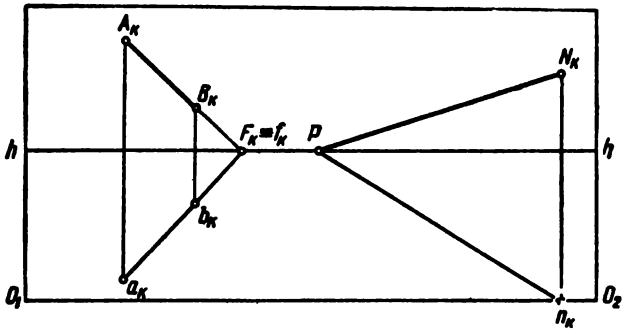

Fig. 344

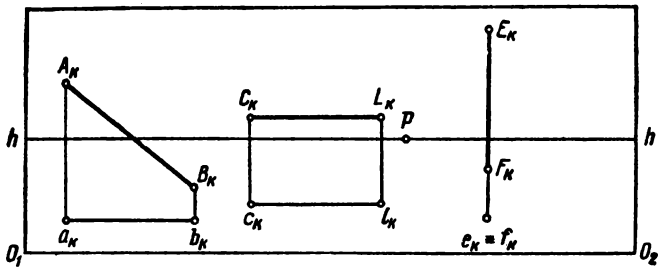

Fig. 345

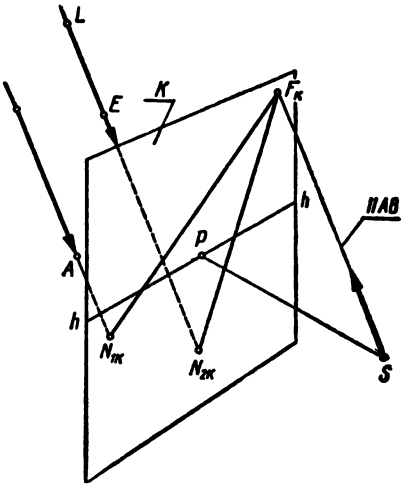

Fig. 346

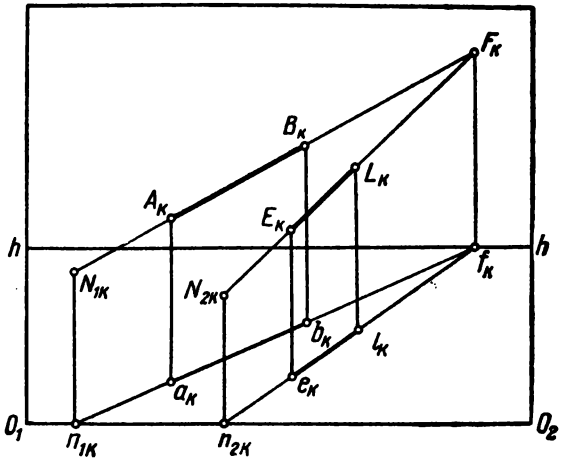

Fig. 347

The perspectives and secondary projections of parallel lines are illustrated in Fig. 347.

When the parallel lines are horizontal, their vanishing point must lie on the horizon.

But if the horizontal lines are perpendicular to the picture, the principal point P will serve as their vanishing point (Fig. 348).

In future, when constructing perspective views of objects, we will often have to construct the perspectives of parallel lines lying in the ground plane T. Their vanishing point lies on the horizon (Fig. 349).

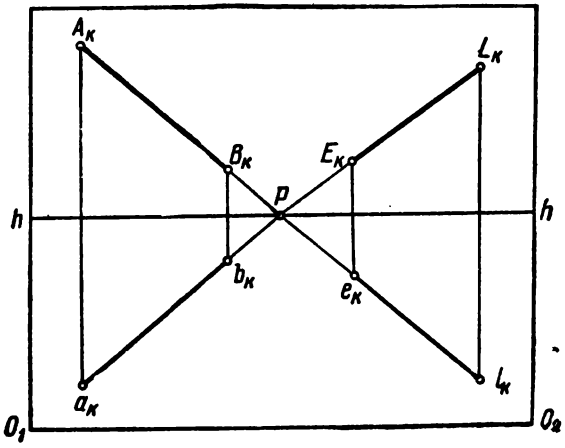

Fig. 348

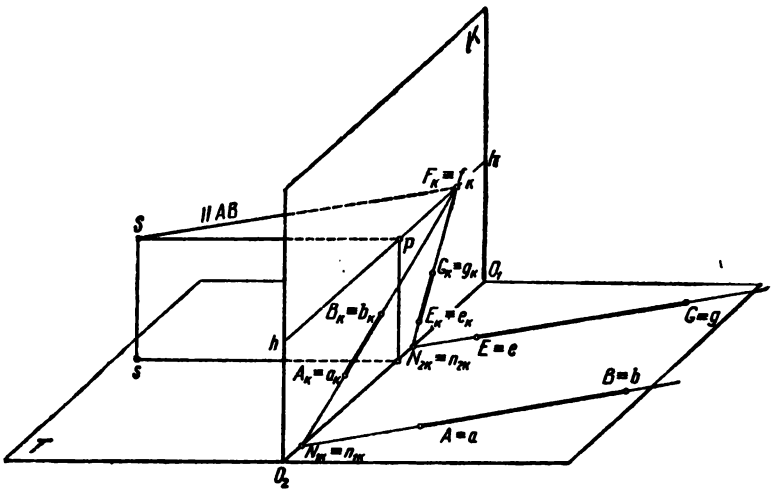

Fig. 349

2. *Intersecting and skew lines.* If two straight lines have a common point, the points of intersection of their perspectives and secondary

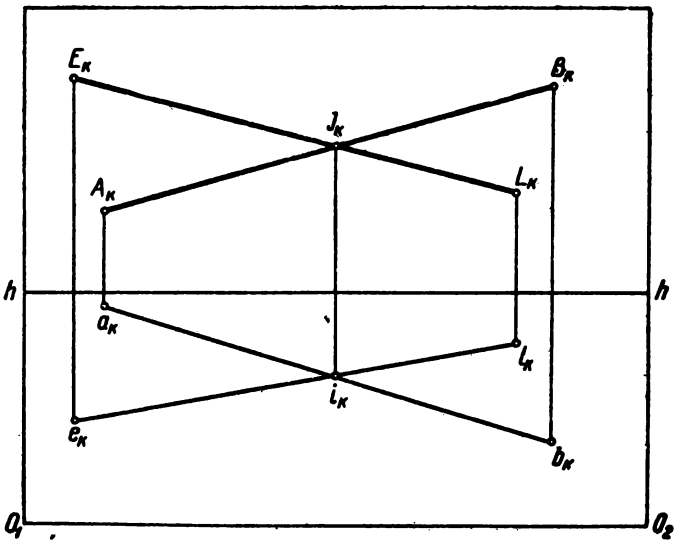

Fig. 350

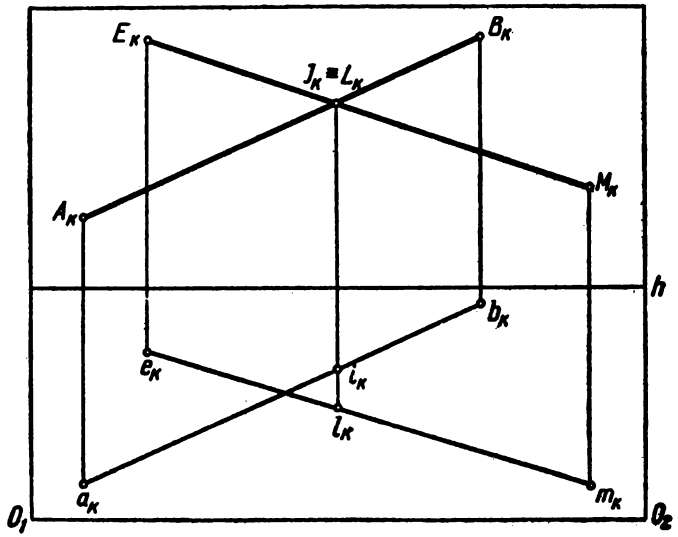

Fig. 351

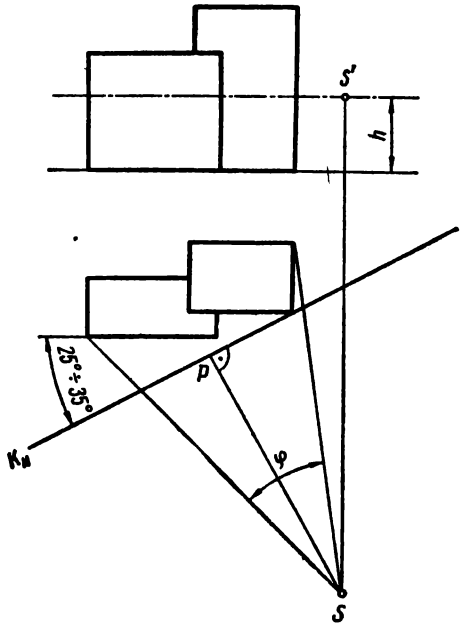

Fig. 352

projections on the picture must lie on a common perpendicular to the line of the horizon (Fig. 350).

Fig. 351 shows two skew lines where the point of intersection of perspectives is related to two different points I and L, the first of which belongs to line AB, the second, to line EM.

Sec. 69. Choice of Point and Angle of View. Orientation of Picture

The following practical rules will help to build suitable perspective views.

1. Take the angle between the projecting rays directed to the extreme points of the plan of the object, the viewing angle φ (Fig. 352) to be taken between the limits from 18° to 53°. The viewing angle has an optimal value of 28°. If the vertical dimensions of an object exceed the length, the viewer should move to a distance of 1 1/2 to 2 heights of the object in order to get the viewing angle in the vertical plane within the permissible limits.

2. The picture plane is oriented so that, firstly, the principal point should lie within the median third of the width of the picture and, secondly, the horizontal trace K_H of the picture plane should make an angle of 25° to 35° with one of the sides of the plan (most frequently with the principal facade). Then the lateral facade will be greatly foreshortened perspectively and the view will, on the whole, be more expressive. It is also advisable to bring the picture plane into coincidence with one of the edges of the object, which will be depicted true size in the perspective projection.

The involved construction of the perspective of an interior is appreciably simplified if the picture plane is placed in a frontal position.

3. The height of the horizon is ordinarily taken at eye level of a person standing on the ground (h=1.5 to 1.7 m). When portraying the construction of a large area, the height of the horizon is taken at 100 m and more. This is known as a "bird's-eye perspective".

Sec. 70. Methods of Constructing Perspective

1. Radial (visual-ray) method. Essentially this method consists in determining the points of intersection of rays with the picture plane.

Fig. 353 illustrates the construction of perspective of a point from its given orthogonal projections.

In addition to the orthogonal projections of point A (a, a'), the orthographic representation shows horizontal trace K_H of the picture

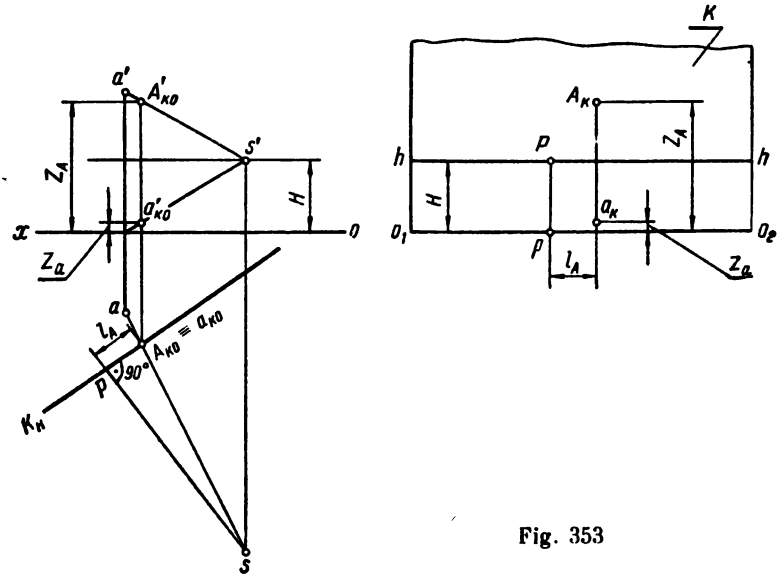

Fig. 353

plane K, the projections of the point of sight S (s, s'), and the principal point of the picture P.

After constructing the projections of ray SA (the horizontal projection sa and the vertical projection $s'a'$), we determine the point of intersection of this ray with the picture plane. The projections of the intersection point thus found are labelled A_{KO} and A'_{KO}.

Next to the orthographic representation is picture K lying in the plane of the drawing; the picture contains the basic elements (ground line, horizon line, and principal point).

It now remains to transfer the point of intersection of the ray with the picture (which was found on the orthographic drawing) to the perspective drawing. Since the position of a point on a plane

is determined by two coordinates, we can take the rectangular coordinates with origin at the foot of the principal point (point p). Then the dimensions of the segments l_A and Z_A taken from the orthographic view yield the perspective A_K of point A. The secondary projection of the point is found by means of segment z_a. This segment determines the coordinate z of the point of intersection of the picture and a ray sent from the point of sight onto the horizontal projection of the point (see orthographic view in Fig. 353).

The radial method is the most elementary method for constructing perspective views in the sense that it may be employed without any knowledge of the theory of perspective. Thus, the object, the point of sight, and the picture plane were all depicted orthographically. In this view we also established the projections of the projecting rays and determined their points of intersection with the picture.

However, the radial method involves repeated use of the same constructions, requires considerable time, and does not ensure sufficient accuracy. Ten elementary operations (in the sense of drawing a line with a ruler and constructing a line-segment of equal length by means of dividers) are needed to construct the perspective of a straight-line segment. The radial method is suitable for constructing the perspective of an object, the plan of which involves many nonparallel lines.

2. Architectural method. Architects frequently employ a method of constructing perspective images by means of the vanishing points of parallel lines. This is known as the architectural method.

Since construction is begun with the secondary projection, the essentials of the method may be demonstrated by constructing the perspective of a figure located in the horizontal plane.

The orthogonal projections of such a figure, which may be regarded as the plan of a building, are shown in Fig. 354. This orthographic view also includes the horizontal trace of the picture plane, the projections of the point of sight and the principal point P.

Noting that the contour lines of the plane may be divided into two pencils of parallel lines, we determine the perspective of points at infinity (F_{1K} and F_{2K}) of each of the pencils, point F_{1K} being the perspective of a point at infinity of the pencil of parallel lines in the direction I_R, and point F_{2K}, in the direction II_R. Both points are found by means of rays SF_{1K} and SF_{2K}, which are respectively parallel to the lines of the directions I_R and II_R. The rays SF_{1K} and SF_{2K}, while parallel to straight lines located in the horizontal plane, will intersect the picture in points lying on the horizon $h\,h$ (Fig. 355). When constructing perspective without enlargement, segments PF_{1K} and PF_{2K} in Fig. 355 are equal to segments PF_{1K} and PF_{2K}, respectively, in Fig. 354.

For the second points in constructing the perspective of each of the straight lines of the contour, it is advisable to employ the characteristic points in which these lines intersect the picture plane, that is, the initial points of the lines. For this purpose, the horizontal projections of all straight lines are extended to intersection with the like trace K_H of the picture. In this way we find points n_1, n_2, n_3,

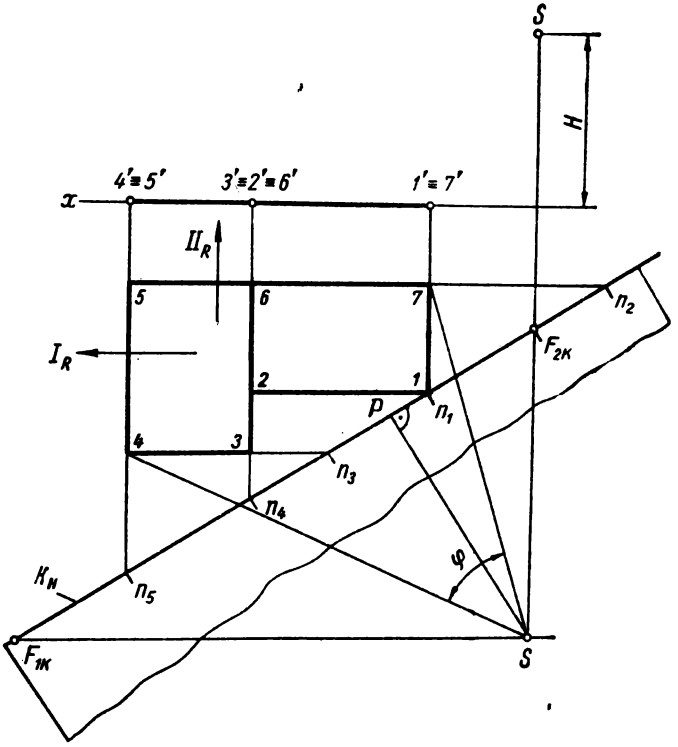

Fig. 354

n_4 and n_5, which lie on the ground line. These points may be transferred from the orthographic view to the picture by means of a strip of paper, the straight edge of which is applied to trace K_H of the picture and the points n_1, n_2, n_3, n_4, n_5 and P marked. Transfer this strip of paper to Fig. 355 and put the edge along the ground line O_1-O_2 of the picture in such a manner that point P on the strip coincides with point p on the ground line.

It remains now to construct the perspective of straight lines, the intersection of which will determine the vertices of the given

contour. Thus, the intersection point of perspectives of lines n_4F_{2K} and n_3F_{1K} represents the perspective of point *III*. The remaining points are found in a similar way. Thus, every point of a plane figure is determined by the intersection of straight lines that belong to two different pencils of parallel lines.

It is to be noted that it is not absolutely necessary to take for the second point the initial points of the lines to construct their perspective. Figs. 356 and 357 show the construction of the perspective of the same contour in the following sequence:

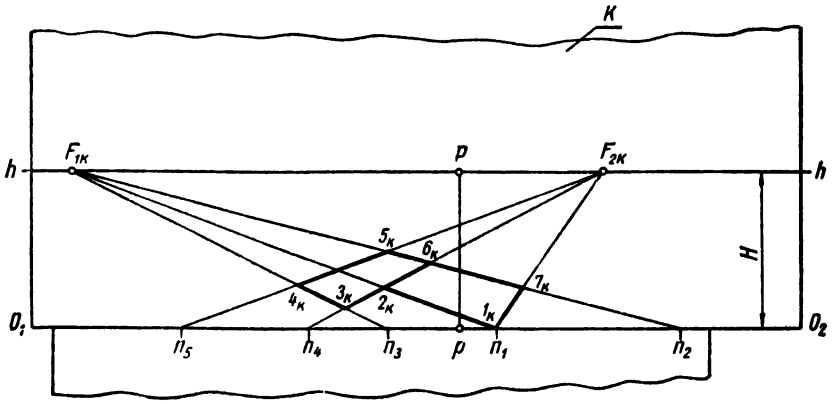

Fig. 355

1. In Fig. 357, on the horizon line of the picture we note the vanishing points F_{1K} and F_{2K} of the pencils of parallel lines in directions I_R and II_R. As before, segments PF_{1K} and PF_{2K} in Fig. 357 are respectively equal to segments PF_{1K} and PF_{2K} in Fig. 356.

2. On the ground line O_1O_2 of the picture (Fig. 357) we determine the position of the perspective of point *I* (*1*, *1'*), through which the picture plane passes. To do this, from the foot of the principal point (point *p*) lay off, to the right, segment $p\text{-}1_0$ equal to segment *p-1* in the orthographic view.

3. Since two lines of different pencils pass through point *I* (*1*, *1'*), we connect its perspective with points F_{1K} and F_{2K}.

4. The next point *II* (*2*, *2'*) is located on line *I-II* of the first direction, whose perspective has already been constructed. To construct the perspective of point *II* (*2*, *2'*), send a ray into it from the point of sight and note the point of intersection of the horizontal projection of this ray with a like trace K_H of the picture, point 2_0.

5. Transfer point 2_0 from the orthographic view to the ground line of the picture. Quite naturally, segment $p\text{-}2_0$ in Fig. 357 is equal to segment $p\text{-}2_0$ in Fig. 356.

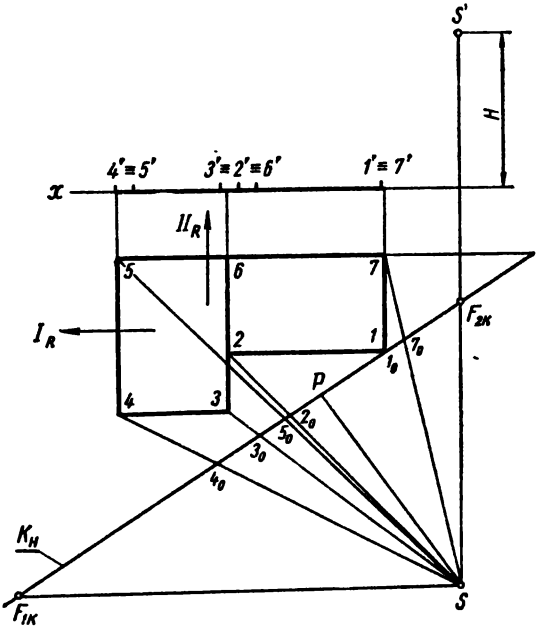

Fig. 356

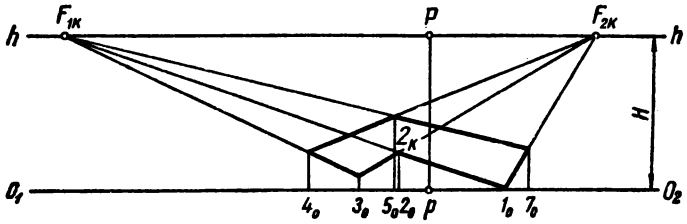

Fig. 357

6. Through point 2_0 draw a vertical line, which, intersecting the earlier constructed line $1_0\text{-}F_{1K}$, determines the position of the perspective of point *II* (2, 2').

The perspectives of the remaining points are obtained in the same sequence.

The above routine for constructing the perspective of a plan has the characteristic feature that we first produce the perspective of a straight line and then, by means of a special ray, we determine a point on it.

This method is advisable for constructing plan perspectives when the points of intersection of some of the lines with the picture lie outside the drawing.

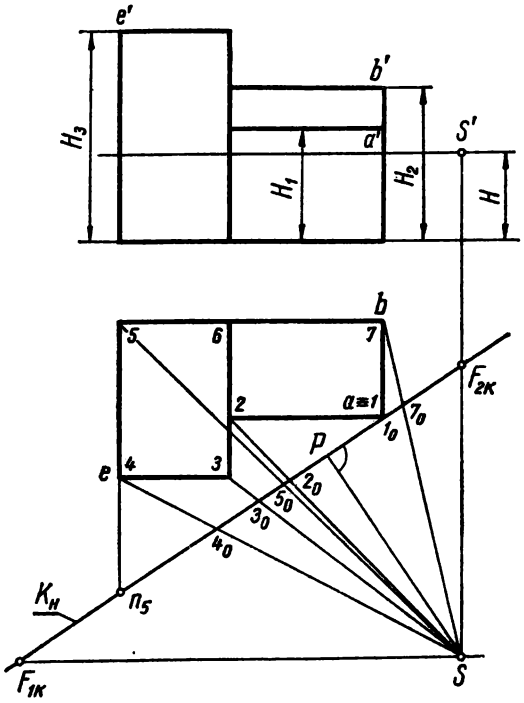

Fig. 358

After constructing the secondary projection, begin depicting the object proper.

Fig. 358 shows the orthogonal projections of two geometric solids, the plan of which is identical with the earlier considered figure. We omit explanations dealing with construction of the secondary projection and describe the process of creating the perspective of the object.

We need only note that in this case the perspective of the object (and its secondary projection) was constructed via a twofold enlargement of all linear dimensions in the picture plane. Which means

that the distance between the ground line of the picture and the line of the horizon in Fig. 359 is not equal to H, as in Fig. 358, but $2H$. Segments PF_{1K} and PF_{2K} in Fig. 359, are twice the size of those segments in the orthographic view, etc.

We refer to Fig. 359 After drawing the secondary projection, construction was carried out in the following sequence:

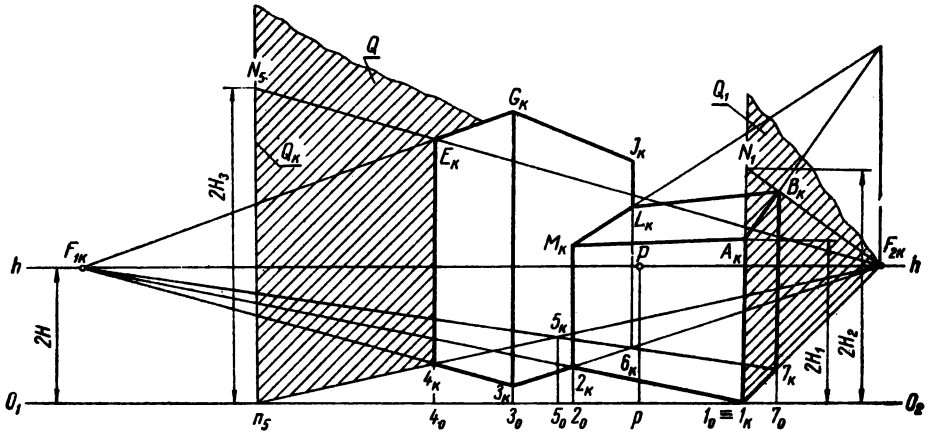

Fig. 359

1. Through all vertices of the secondary projection (points 1_K, 2_K, 3_K, etc.) draw vertical straight lines.

2. From point 1_K, on the vertical line passing through it, lay off segment $1\text{-}A_K$ of length $2H_1$, since this segment lies in the picture plane and the scale is 2 : 1.

3. Through point A_K draw a straight line to the vanishing point F_{1K} bounding it by a vertical line that passes through point 2_K. We obtain point M_K.

4. In order to produce the perspectives of the vertical edges of length H_3 passing through points 3_K, 4_K, 5_K, 6_K, through any edge pass a vertical plane Q and construct the line of intersection of Q with the picture plane K. Then, laying off on this line (Q_K) from the ground line of the picture segment n_5N_5, equal to the given altitude H_3 (with account taken of the scale), draw in plane Q a horizontal line of the given level H_3 to intersection with the perspective of the given edge.

Thus, in Fig. 359, plane Q is passed through the vertical line that determines point 4_K. In this case, Q coincides with the rear left face of the object. Line n_5F_{2K} is the horizontal trace of plane Q (the line of intersection of Q and T), while line Q_K is the line of intersection

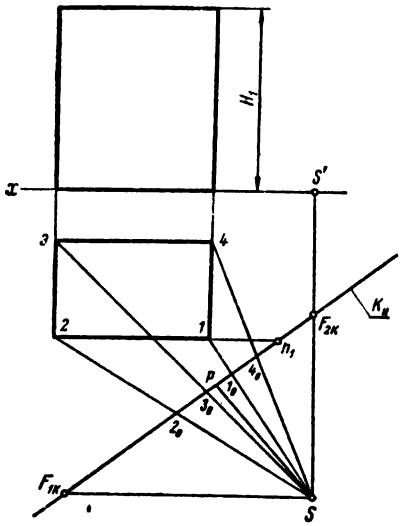

Fig. 360

of plane Q and the picture. On this line Q_K from point n_5 lay off n_5N_5 (the height of the edge, H_3; in this case, take account of the scale $2H_3$). Joining points N_5 and F_{2K}, we get the perspective of the horizontal line of the level: H_3. The intersection of the latter with the vertical line passing through point 4_K determines point E_K and completes the process of constructing the perspective of edge 4_K-E_K external to the picture plane.

5. By alternate use of the left and right vanishing points, we draw the upper horizontal edges E_KG_K and G_KI_K of the visible faces of the rectangular parallelepiped.

6. Repeating the construction given in Item 4, construct the perspective of edge 7_K-B_K.

7. Joining points B_K and F_{1K}, we get the perspective of the horizontal edge B_KL_K.

8. Draw the perspectives of two parallel inclined straight lines A_KB_K and M_KL_K.

A check on the accuracy of construction is the convergence of lines A_KB_K and M_KL_K on the vertical line drawn through the vanishing point F_{2K}.

A particular case is when the height of the horizon is equal to zero or is so small that the secondary projection of the object is extremely compressed. We illustrate the construction of the perspective of a rectangular parallelepiped by means of a plan recommended for such cases: the depressed plan.

A peculiarity of the transition from orthogonal projections (Fig. 360) to the perspective view (Fig. 361) is that the secondary projection of the object is established not on the ground plane, which in this case coincides with the plane of the horizon, but on a certain auxiliary horizontal plane displaced downwards from the plane of the horizon an arbitrary distance h'.

Line O_3O_4 in Fig. 361 is parallel to the line of the horizon and is the line of intersection of the auxiliary plane with the picture plane; it is generally called the depressed ground line of the picture.

Without repeating the earlier described constructions of the secondary projection and the perspective of the object, we note that in producing the perspectives of the vertical edges of the parallelepiped, their altitudes (equal to H_1) are laid off not from the depressed ground line but from the line of the horizon, which in this case coincides with the true ground line of the picture: line O_1 and O_2.

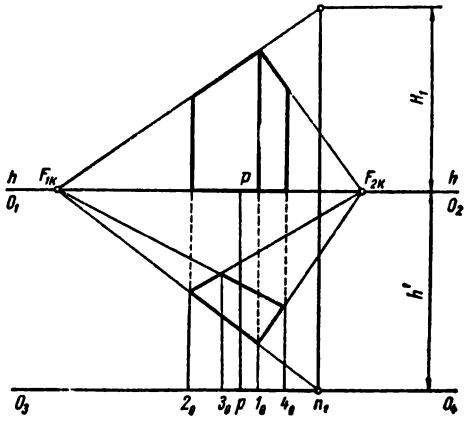

Fig. 361

The architectural method, which ensures considerable accuracy of perspective views and—what is most important — is very time-saving, does not preclude the radial method. While the architectural method is used for constructing the perspective of the object as a whole, certain points may be constructed via the radial method.

The architectural method involves certain difficulties only when one or both of the vanishing points F_{1K} and F_{2K} of the pencils of parallel lines lie beyond the drawing board.

If the size of the board allows for only one vanishing point (say F_{2K}), then it is best to determine each point of the secondary projection by the intersection of two straight lines, the first of which belongs to the pencil with the vanishing point F_{2K}, while the latter is a straight line of any other pencil of horizontal parallel lines. The only requirement is that the direction of the second pencil should be such that its vanishing point lies within the working space. This is usually a pencil of lines perpendicular to the picture, the vanishing point of which pencil lies in the principal point of the picture. As before, all lines of the first and second pencils are determined by means of two characteristic points: the initial point and a point at infinity. This is done in Figs. 362 and 363 in the construction of the perspective of a schematized shape of a building.

Fig. 364 illustrates the transition from an orthogonal drawing to a perspective drawing in the case of point A.

First of all, through the horizontal projection of this point draw straight lines an_1 parallel to sF_{2K} and an_2 perpendicular to K (see Fig. 362), the perspectives of which are constructed by means of the initial points n_1 and n_2 and the infinitely distant points F_{2K}

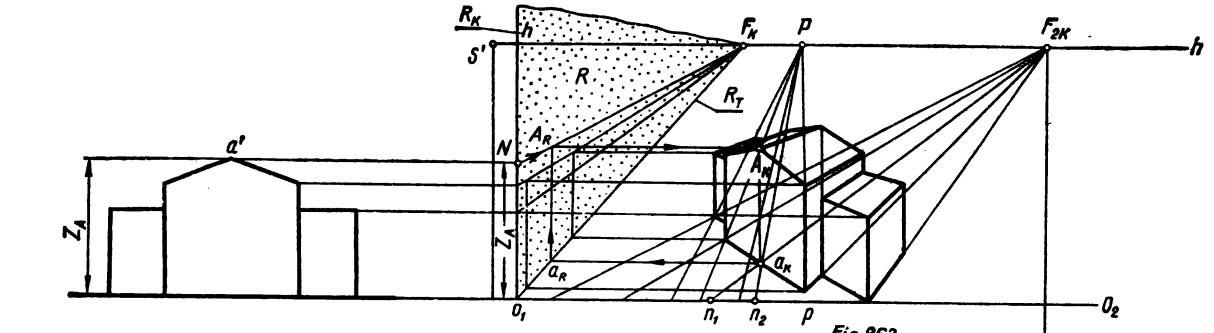

Fig. 363

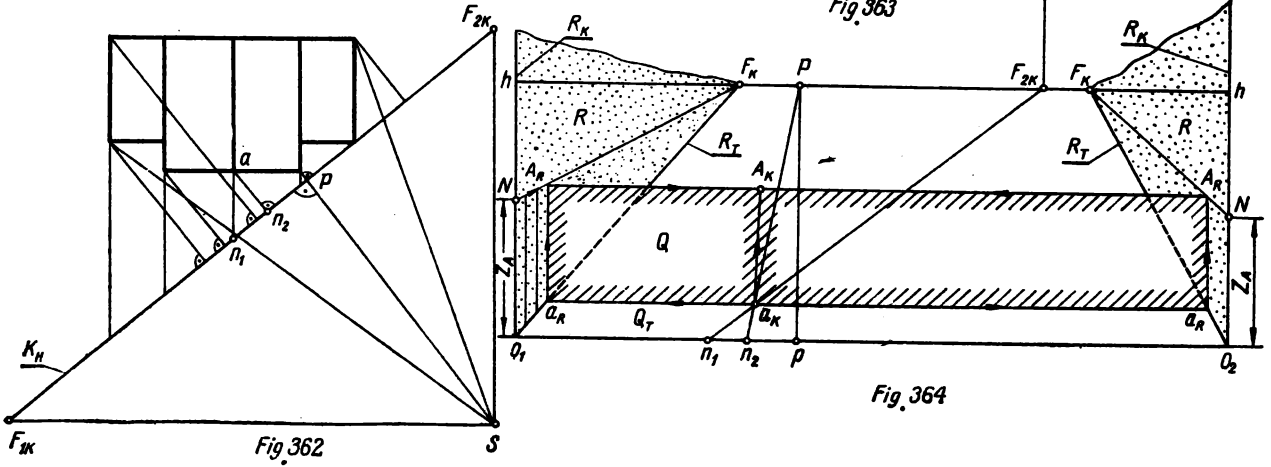

Fig. 362

Fig. 364

and P. The intersection of these straight lines determines the secondary projection of the given point.

To construct the perspective of the point, take advantage of the vertical plane R, which intersects the picture plane and the ground plane along straight lines R_K and R_1 respectively. To relieve the sheet (reserved for perspective) of an auxiliary grid of straight lines, put plane R some distance away from the principal point, as shown in Fig. 364. Naturally, for practical purposes we need only one of the two planes shown in the drawing. Point F_K of plane R is the perspective of an infinite point of the horizontal lines of this plane. (If F_K coincided with P, then plane R would be perpendicular to the picture plane.)

In plane R draw one of its horizontal lines NF_K, all points of which have $z=Z_A$.

Of all possible vertical segments contained between straight lines NF_K and R_1, we must find the one that is as distant from the picture plane as point a_K.

This segment a_RA_R is determined with the aid of plane Q, which is parallel to the picture plane and passes through point a_K. We conclude by constructing the vertical segment

$$a_KA_K = a_RA_R$$

Analogous constructions for the remaining points of the building are illustrated in Fig. 363.

When both vanishing points (F_{1K} and F_{2K}) lie beyond the limits of the drawing and the working area, it is best to employ the so-called method of scales.

3. Method of scales. The essentials of this method are illustrated in the construction of the perspective of a point. Point A is referred to a rectangular coordinate system $OXYZ$ located as shown in Fig. 365: the origin lies on the ground line of the picture, the coordinate plane OXZ coincides with the picture plane, and the y-axis is perpendicular to the picture plane.

The orthogonal projections of the coordinate axes are shown in Fig. 366. Also labelled are the coordinates of point A (x_A, y_A, z_A), which are used in Fig. 367 to determine its perspective.

First note that the origin in Fig. 366 is chosen at an arbitrary distance from point P. But when constructing the perspective of the system of rectangular coordinates (Fig. 367), point O_K is marked on the ground line of the picture at a distance from p equal to the length of the segment OP in Fig. 366.

The x- and z-axes are shown true size in the picture plane at right angles to one another. However, the perspective of the y-axis, as a line perpendicular to the picture plane, must pass through the principal point P, which, as has been demonstrated earlier, is the

perspective of a point at infinity of straight lines perpendicular to the picture plane. Axes constructed in perspective are called: the scale of widths (axis $O_K x$), the scale of depths (axis $O_K y$), and the scale of heights (axis $O_K z$).

The perspective of a point is constructed from its coordinates according to the following plan:

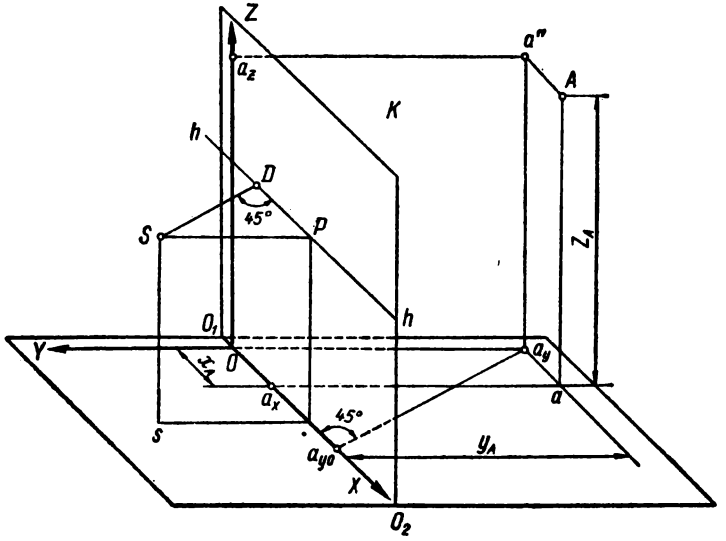

Fig. 365

1. In Fig. 367, lay off the abscissa x_A on the scale of widths and draw a line $a_{KX}P$ as the perspective of the straight line $a_X a$ (see Fig. 366). This line must contain the secondary projection a_K of point A.

2. In order to construct point a_y on the scale of depths, take advantage of line $a_y a_{yO}$, which makes a 45° angle with the y-axis and, hence, with the picture plane (see Fig. 366). It will readily be seen that the vanishing point of such a line is the point on the horizon a distance from the principal point P of the picture equal to the principal distance, which is the distance of the point of sight S from the picture plane.

Indeed, a consideration of the right isosceles triangle sPD (see Fig. 366) shows that the horizontal ray sD drawn at a 45° angle to the picture plane and thus determining the perspective of an infinite point of the pencil of parallel lines intersects the picture plane in the point of distance D.

Point $a_y a_{yO}$ is the beginning of the straight line a_{yO} under consideration and has to be plotted on the scale of widths, utilizing ordinate y_A of point A.

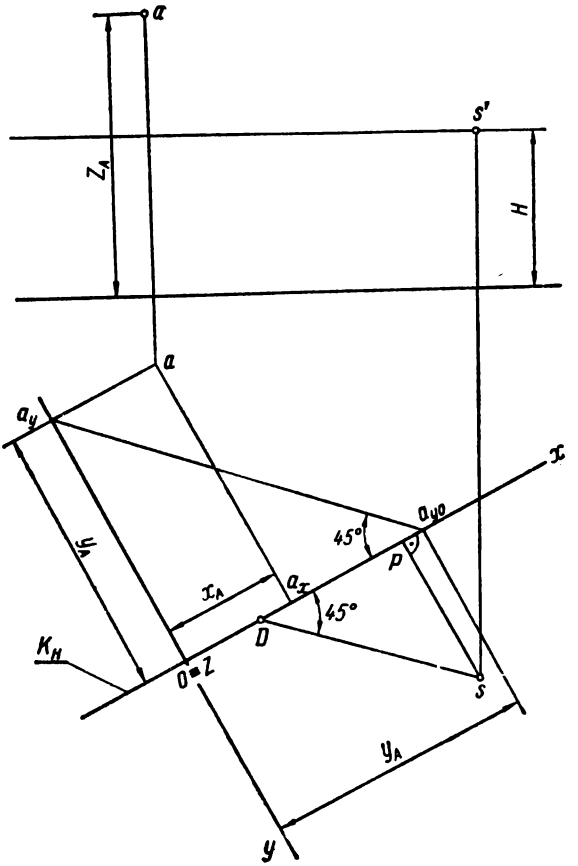

Fig. 366

Connecting this point with D, we construct the perspective of a straight line, the perspective intersecting the scale of depths in the point a_{Ky}.

3. Since points a_y and a are located on a single straight line parallel to the picture plane (see Fig. 366), the secondary projection of the point will be located at the intersection of the earlier constructed line $a_{KX}P$ with the straight line $a_{Ky}a_K$, which is parallel to the ground line.

4. On the scale of heights (see Fig. 367) lay off a segment equal to z_A (the z-coordinate of point A) and join point a_{Kz} with P. The vertical line from a_{Ky} drawn to intersection with line $a_{Kz}P$ will yield segment $a_{Ky}a_K$, which in perspective is equal to the coordinate z_K.

Finally, on the vertical line passing through the secondary projection of point A we have to construct a segment equal to $a_{Ky}a''_K$. This is done with the aid of the horizontal line $a''_K A_K$.

Even when employing the method of scales in constructing a perspective image, the picture may be enlarged; that is, all linear

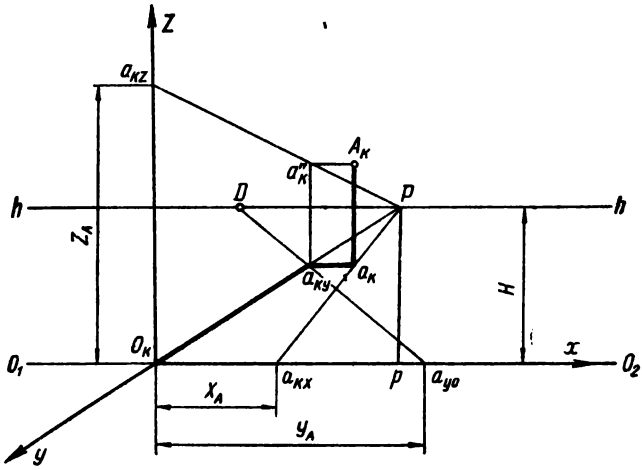

Fig. 367

dimensions taken from the orthogonal drawing are increased by a factor of n. In the process, certain points, say D or a_{yO}, may prove inaccessible. Then we resort to what are known as fractional points of distance, which in Fig. 368 are labelled D_2, D_3, D_n. Each of these points is located on the horizon line and is at a distance $\delta/2$, $\delta/3$ and δ/n (δ is the principal distance and n is any number), respectively, from the principal point P.

Suppose a lack of space has forced us to make use of a fractional point of distance D_2. Then, on the scale of widths, we lay off segments of length half the y-coordinate.

The intersection of line $2\text{-}D_2$ and the scale of depths defines the same point a_{Ky}, which could have been obtained with the aid of straight lines $D\text{-}a_{yO}$, $D_3\text{-}3$, or $D_n\text{-}n$. This is due to a proportionality of segments as expressed by the following equalities

$$\frac{O_K a_{Ky}}{a_{Ky}P} = \frac{y_A}{\delta} = \frac{y_A/n}{\delta/n}$$

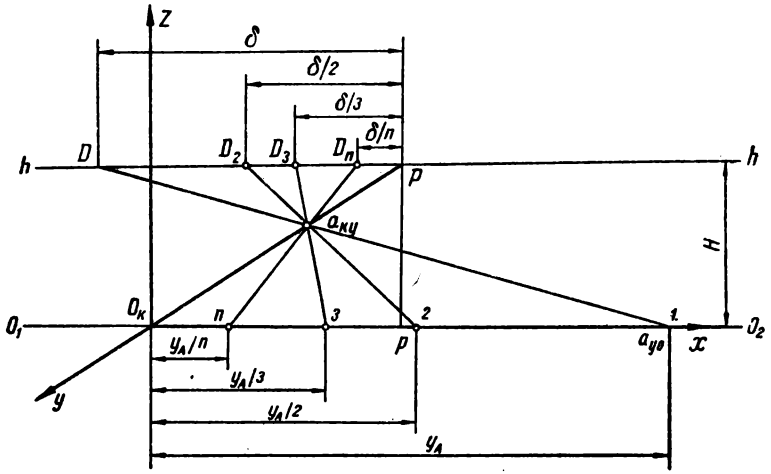

Fig. 368

Sec. 71. Examples in the Construction of Perspective

The examples given in this section illustrate certain peculiarities of construction that one frequently encounters in perspective representations of plane figures and three-dimensional solids. In many

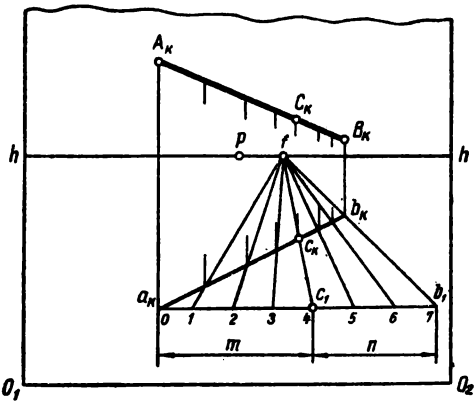

Fig. 369

cases we make use of the perspective division of a segment in a given ratio. We refer to Fig. 369, where segment A_KB_K is divided in the ratio $m : n$. The construction is based on the fact that to equal segments of a straight line parallel to the picture plane there correspond

equal segments on the perspective of the same line. First divide the secondary projection of the segment in the ratio $m : n$. To do this, through one of its extremities draw a line parallel to the ground line of the picture and on it lay off from point a_K segments that are proportional to m and n (in Fig. 369 $a_K c_1 = m$ and $c_1 b_1 = n$). Then join points b_1 and b_K and extend $b_1 b_K$ to intersection with the horizon in point f.

The desired point c_K on the secondary projection $a_K b_K$ is determined by means of the straight line fc_1, since lines $b_1 b_K$ and $c_1 c_K$ are

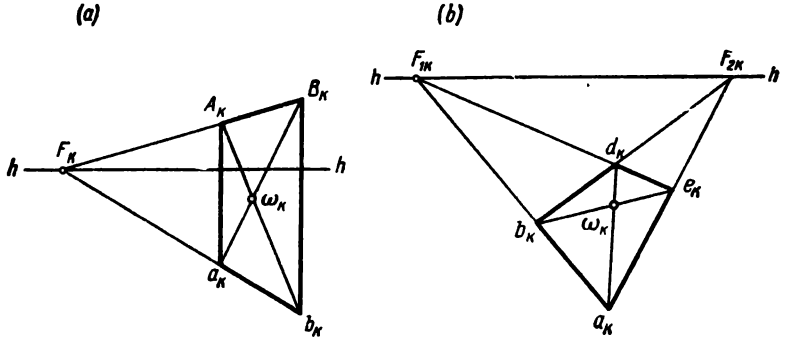

Fig. 370

the perspectives of parallel lines lying in the ground plane. It now remains to find the perspective of point c_K from the secondary projection of the point. This is done by means of the vertical line $c_K C_K$. Fig. 369 also shows the perspective division of the segment into equal parts (7 parts).

Example 1. *Determining the centre of symmetry of a rectangle.*

Fig. 370 depicts a rectangle in the vertical (*a*) and horizontal (*b*) planes. The desired centre (point ω_K) is located at the point of intersection of the diagonals.

Example 2. *The perspective of a circle.*

Fig. 371 shows a circle located in the ground plane. To start with we construct the perspective of a square whose sides are parallel and perpendicular to the picture. Of 8 points shown in Fig. 371, the even points are located at the midpoints of the sides of the square, the odd, on the diagonals. It will be noted that the midpoints of the sides perpendicular to the picture are defined with the aid of a straight line that passes through the centre of symmetry of the square: point ω_K. The perspective of the circle has linear dimensions enlarged by a factor of two.

The construction is similar for a circle located in a vertical plane. The quadrangle $a_K A_K B_K b_K$ in Fig. 372 is the perspective of half a square in which it is necessary to inscribe a semicircular arch (semicircle).

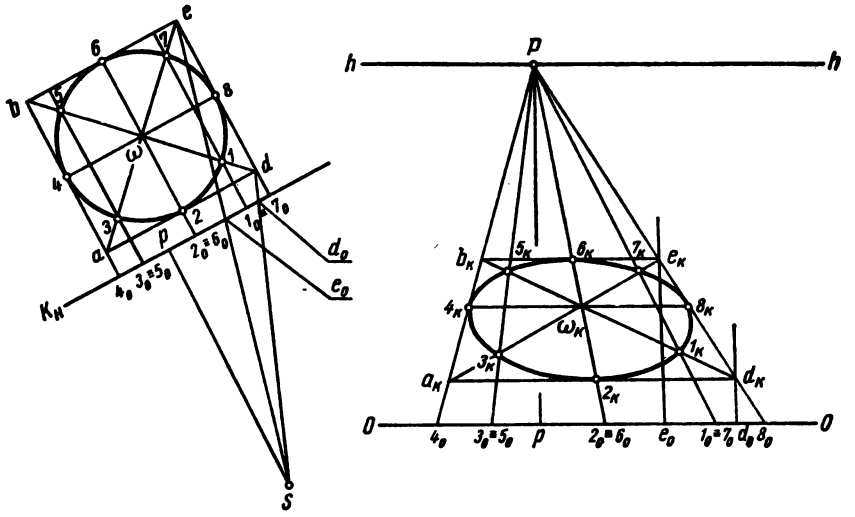

Fig. 371

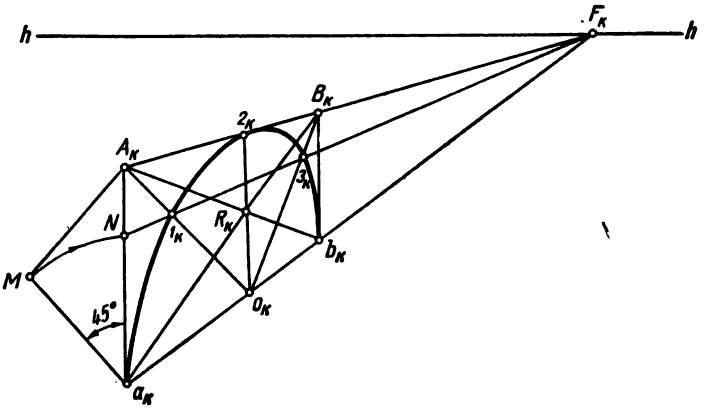

Fig. 372

As in the preceding case, the centre of this arch (point O_K) and the midpoint of side $A_K B_K$ (point 2_K) are obtained by means of a straight line that passes through the centre of symmetry of the quadrangle (point R_K).

Points 1_K and 3_K, in which the desired circle intersects the semidiagonals of the square (O_KA_K and O_KB_K) are constructed by perspective division of the segments in a given ratio.

Since the ratio of the diameter of a circle inscribed in a square to a diagonal of the square is $1 : \sqrt{2} \approx 7 : 10$, this will be the ratio of the radius of the circle to the semidiagonal of the square.

First, the vertical segment $A_Ka_K\left(\frac{a_KN}{a_KA_K}=\frac{1}{\sqrt{2}}\right)$ is divided in the given ratio by means of the right isosceles triangle A_KMa_K. Then the straight line NF_K which is parallel to A_KB_K, will divide the semidiagonals of the square in the same ratio.

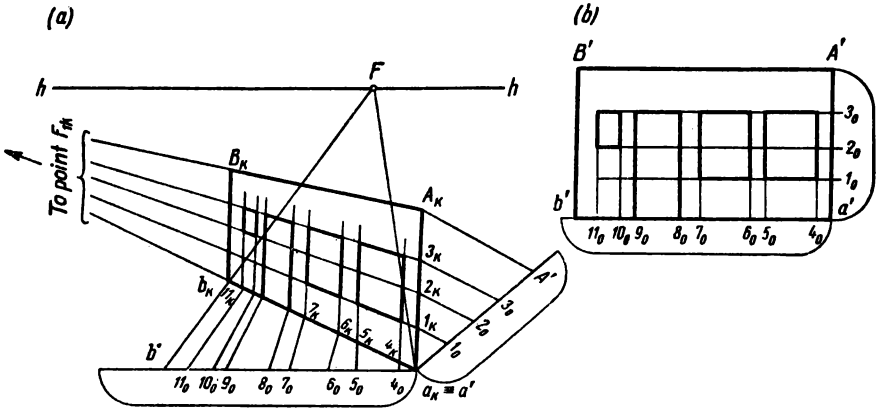

Fig. 373

Example 3. *The construction of window and door openings.*

The construction of window and door openings is greatly simplified if the problem is reduced to dividing the perspectives of a segment into proportional parts. On the orthogonal projection of the front of a building (Fig. 373*b*) make a grid of horizontal and vertical lines passing through the angles of the contours of the windows and doors. This grid may be constructed on the picture plane as follows:

1. Apply a strip of paper to segment $a'A'$ (on the orthogonal projection of the facade of the building) and on its edge mark the points a', 1_0, 2_0, 3_0, and A', through which the horizontal lines of the grid pass.

2. Transfer the strip of paper to the picture plane so that it makes an arbitrary angle with the straight line a_KA_K and point a' coincides with point a_K.

3. Through points 1_0, 2_0 and 3_0 on the perspective of the facade of the building (Fig. 373*a*) draw straight lines parallel to $A'A_K$ to intersect with a_KA_K.

4. Joining points 1_K, 2_K and 3_K with the left vanishing point F_{1K}, we get the perspectives of the horizontal lines of the grid.

5. Construction of the vertical lines of the grid requires a proportional division of the perspective of segment $a_K b_K$. To do this, draw the straight line $a_K b'$ parallel to $h\,h$ and transfer to it (with the aid of the strip of paper) points 4_0, 5_0, 6_0, ... 11_0, b'.

Extending the line $b'b_K$ to intersection with the line of the horizon determines the position of the perspective of an infinitely distant point of the pencil of parallel lines, with the aid of which we obtained, on segment $a_K b_K$, points 4_K, 5_K, 6_K, ... 11_K.

The vertical lines of the grid will pass through the points thus found and, intersecting the corresponding horizontal lines, will determine the contours of the window and door openings.

Example 4. *Constructing the perspective of a reinforced concrete span-arch structure.*

Fig. 374 illustrates the orthogonal projections of a bridge. The same figure shows the horizontal trace of the picture plane (K_H). The limited size of the drawing did not permit indicating the position of the point of sight, the principal point, and the vanishing points of pencils of parallel lines. The outer contour of the secondary projection on the depressed plane and the perspective of the bridge at horizon height zero are shown in Fig. 375.

The information needed to execute these constructions is given in Sec. 70.

We shall point out certain peculiarities in the construction of the perspective of an arch-span structure, the contour of which is located in the vertical plane Q.

In Figs. 374 and 375, the horizontal trace of this plane, Q_T, is shown as a dashed line. Through point $\bar{n}_0$ on the depressed ground line of the picture plane (line O_3O_4) draw a vertical straight line Q_K, which is the line of intersection of Q and K. The perspective thus constructed of a grid of horizontal and vertical lines enables us to determine the contour points of the arch.

We omit a detailed description for constructing the perspective of the grid (this was given in Example 3) and note that the elevations of points H_1, H_m, H_a, H_b, etc., were marked on the line Q_K from point n_0 (from the horizon).

To construct the arch contour located in plane Q_1, we can take advantage of the earlier found points in plane Q.

Thus, in Fig. 375, points C_K and R_K, which lie in plane Q_1, were obtained with the aid of rays drawn from the vanishing point F_{2K} through points A_K and M_K that belong to plane Q.

On these rays we determine the positions of points C_K and R_K by vertical straight lines that pass through the secondary projections of the points (c_K and r_K).

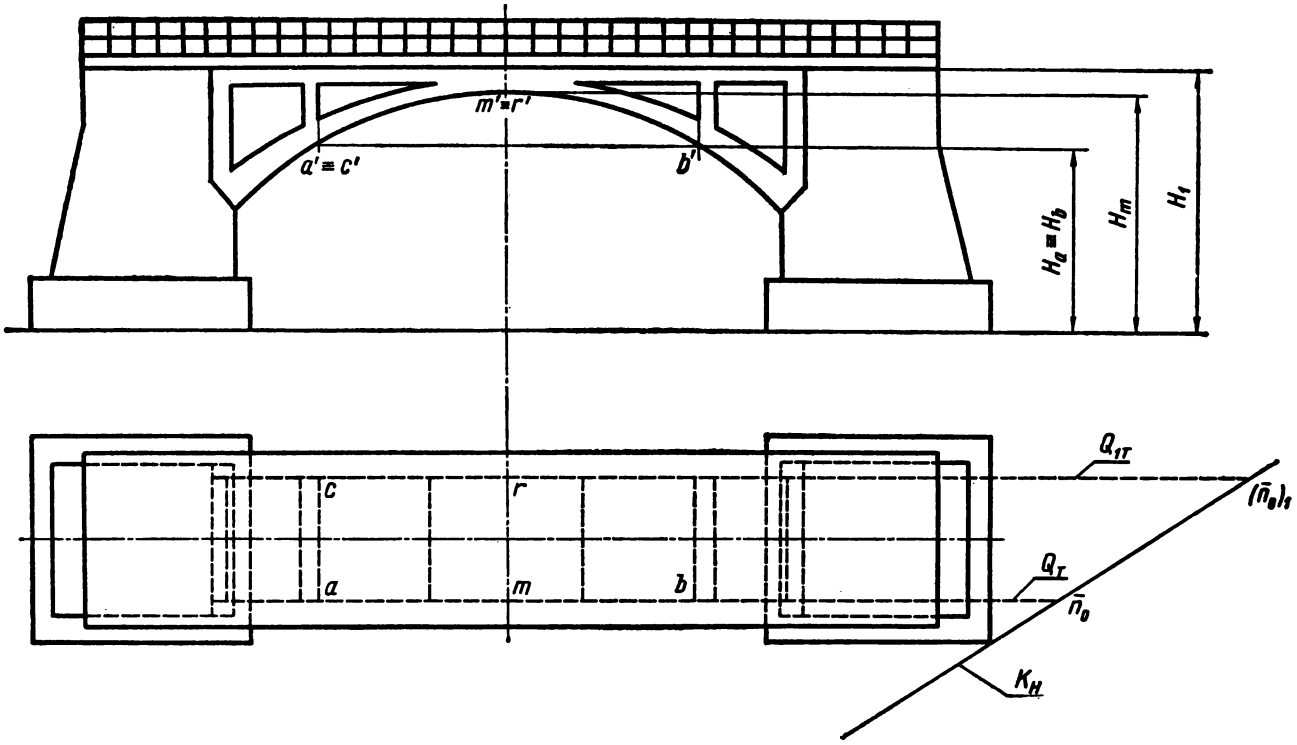

Fig. 374

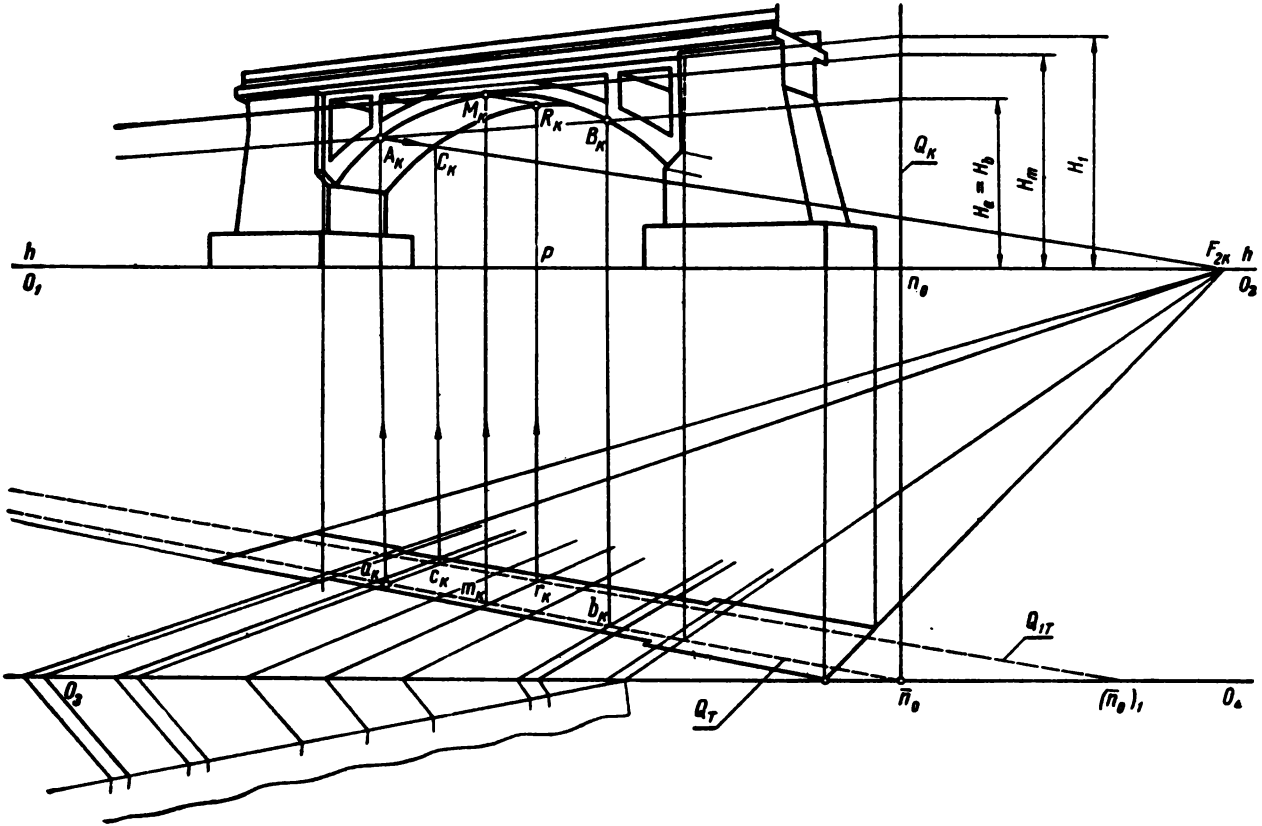
$M_к$
$R_к$
$A_к$
$C_к$
$B_к$
$Q_к$
$H_a = H_b$
H_m
H_1
h
O_1
p
n_0
$F_{2к}$
h
O_2
$a_к$
$c_к$
$m_к$
$r_к$
$b_к$
Q_{1T}
O_3
Q_T
$\bar{n}_0$
$(\bar{n}_0)_1$
O_4

Fig. 375

Sec. 72. Metrical Problems of Linear Perspective

Let us examine the solution of certain metrical problems.

Problem 1. To determine the true length of segment AB, the perspective and secondary projection of which are given in Fig. 376.

Let us first consider the solution of this problem in Fig. 377, which gives a pictorial representation of segment AB and its horizontal projection ab in a system of planes of linear perspective. To determine the required length, make segment AB coincide with the picture plane, taking line $I\text{-}I_1$ as the axis of revolution. This axis must lie in the picture plane and pass through point N_K (the beginning of AB) in which the extended segment intersects K. With the axis in a vertical position, the extremities of segment AB will move in arcs of horizontal circles. In the ground plane, the arcs of circles will also describe the horizontal projections of the extremities of the segment.

Since the angular displacements of the extremities of AB are equal, the chords AA_0 and BB_0 will be parallel to one another. And since each is horizontal, their projections aa_0 and bb_0 will also be parallel to them.

To construct the perspectives of parallel chords, it is necessary to determine their common vanishing point F_K. The latter is found with the aid of ray SF_K, which is parallel to chords AA_0 and BB_0. To construct point F_K on the picture plane, we take advantage of the fact that line-segment SF_K is the base of the isosceles triangle SF_Kf_1, the vertex f_1 of which is the secondary projection of the infinite point of the given line-segment AB. Indeed, in Fig. 378 we have a top view of the system of planes of linear perspective. We consider triangles aa_0n_K and sF_Kf_1. Since the sides of the second one are parallel to the corresponding sides of the first, they are similar. But triangle aa_0n_K is an isosceles triangle ($n_Ka=n_Ka_0$), and therefore the second triangle sF_Kf_1 will also be an isosceles triangle. Bring this triangle into coincidence with the picture plane by revolving it about the horizon line that contains vertices f_1 and F_K. The former is determined by the intersection of the secondary projection of line-segment a_Kb_K with the line of the horizon (see Fig. 376; the subsequent explanations will refer to this figure). The second point is the required one. The point of sight S_C (coincident with the picture plane) will be seen to lie on the perpendicular to the horizon; and segment S_CP will be equal to the principal distance which is assumed to be given. Striking an arc of radius f_1S_C from point f_1 as centre, we get on the line of the horizon the vanishing point of parallel chords: point F_K. Constructing the perspectives of these chords (F_KA_K and F_KB_K) and their secondary projections (F_Ka_K and

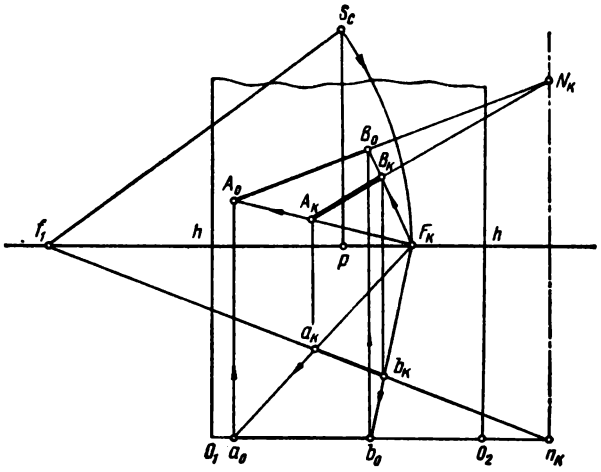

Fig. 376

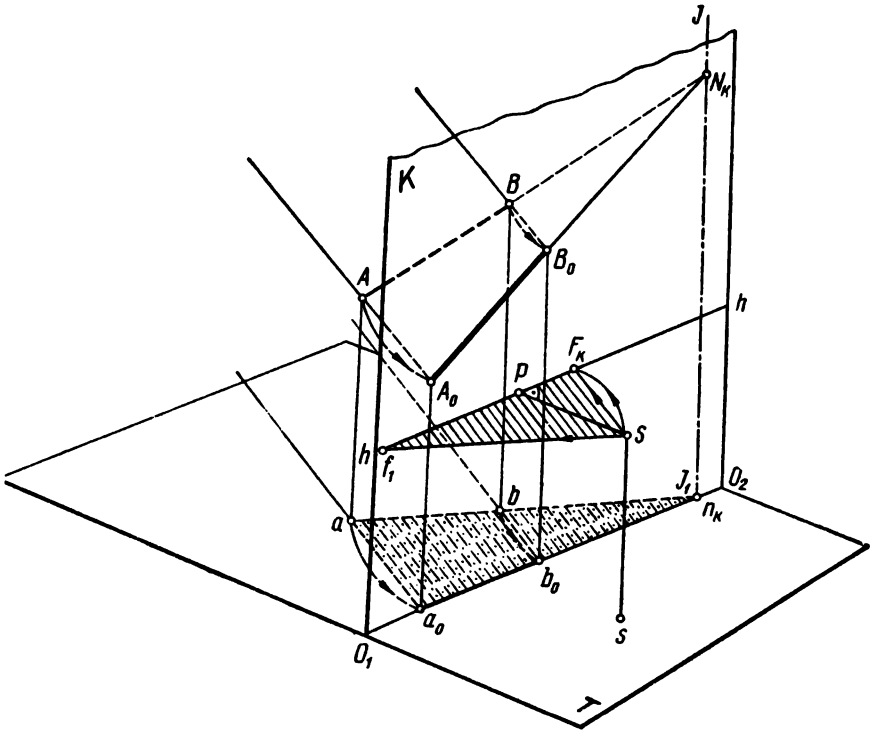

Fig. 377

$F_K b_K$), we find points A_0 and B_0, in which the chords intersect the picture plane (the initial points of the chords). Segment A_0B_0 will be the required one. Chords AA_0 and BB_0 (see Fig. 377) are sometimes called lines of equal sections, since they intersect both the given segment and the picture plane in points separated by equal distances ($AB=A_0B_0$).

Problem 2. To determine the true size of a linear angle BAE (Figs. 379 and 380).

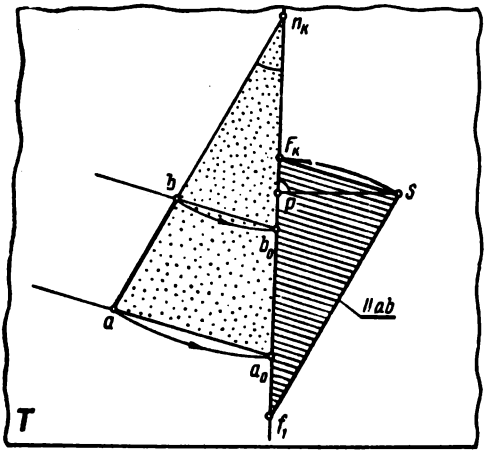

Fig. 378

Draw from the point of sight S two rays SF_K and SF_{1K}, which are respectively parallel to the sides of the angle (see Fig. 379). It is obvious that the angle between the rays will be equal to the given angle. We bring into coincidence the plane of angle F_KSF_{1K} with the picture plane and take line F_KF_{1K} (points F_K and F_{1K} are the perspectives of infinite points of the sides of the given angle) for the axis of revolution.

Then the orthogonal projection of the vertex of the angle between the rays (point S) on the picture plane will move along the perpendicular to F_KF_{1K}. At the instant of coincidence with the picture plane, point S will be separated from line F_KF_{1K} by a distance equal to the radius of revolution. The problem reduces to determining this radius. The required radius R_S is the hypotenuse of a right triangle whose sides are the principal distance SP and segment PM_K perpendicular to the axis of revolution. Fig. 380 shows just such a triangle M_KPS_1. Side PS_1 is equal to the principal distance given in the picture plane by the segment PD (D is the point of distance).

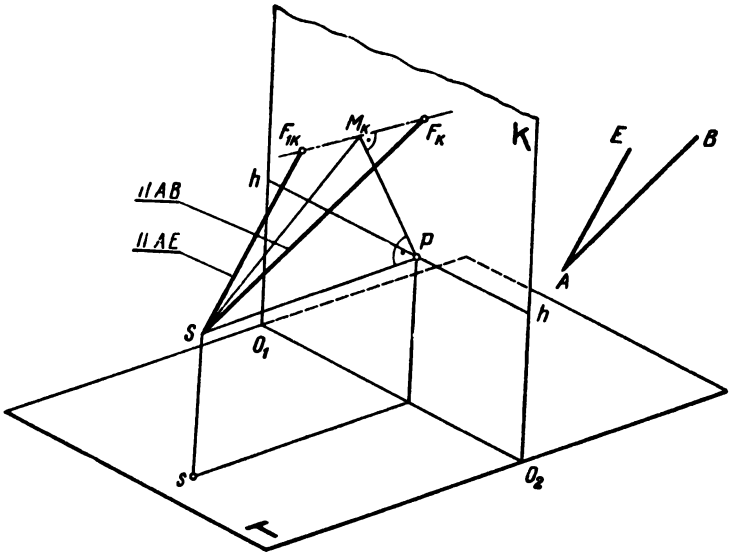

Fig. 379

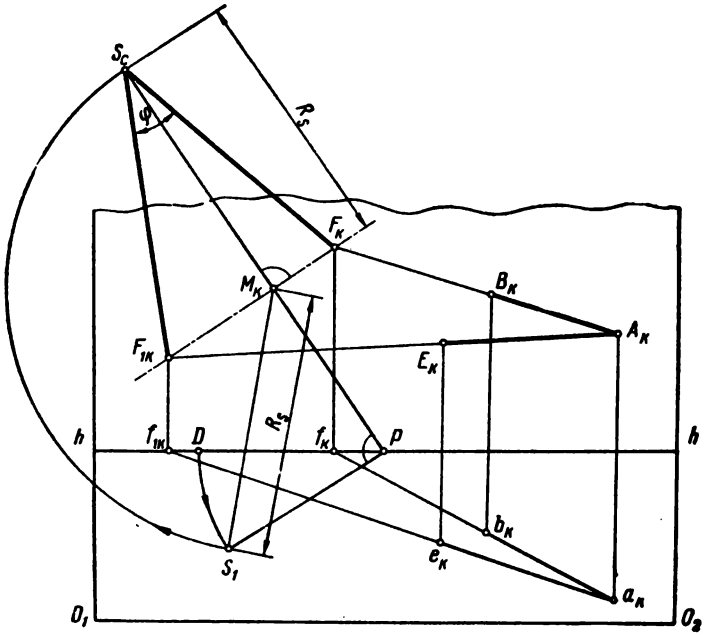

Fig. 380

In Fig. 380 the vertex (S_C) of the angle coincident with the picture plane is determined by intersection of the straight line PM_K with an arc of radius R_S struck from point M_K, the centre of the arc.

Problem 3. To construct the perspective of a square lying in the ground plane T if the perspective of one side (A_KB_K) is given (Fig. 381).

To solve the problem we must first determine the vanishing points of the opposite sides of the square. Both points F_K and F_{1K} must lie on the horizon. To find the first one, extend the given segment A_KB_K to intersect the horizon. To construct the second vanishing

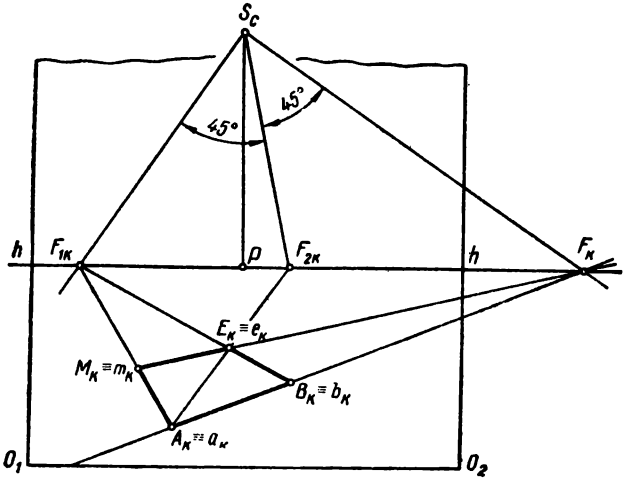

Fig. 381

point, bring the point of sight S into coincidence with the picture plane and draw through it two mutually perpendicular lines S_CF_K and S_CF_{1K}. Both straight lines may be regarded as rays lying in the picture plane and extending from the point of sight S to infinitely distant points of the sides of the square that also intersect at right angles. The point F_{1K} thus found enables us to construct the perspectives of straight lines perpendicular to A_KB_K. It remains to determine the third vertex of the square on one of the constructed lines. This is done by drawing the bisector S_CF_{2K} of the right angle $F_KS_CF_{1K}$, which should be regarded as a ray coincident with the picture plane and directed from the point of sight S parallel to the diagonal of the square that passes through vertex A_K of the constructed right angle. This ray (the bisector of the right angle) intersects the horizon in point F_{2K}. This latter point is the perspective of the infinite point

of the diagonal of the square. The diagonal has enabled us to find the third vertex of the square: point E_K. Intersection of lines $A_K F_{1K}$ and $E_K F_K$ yields the fourth vertex, M_K, of the desired figure.

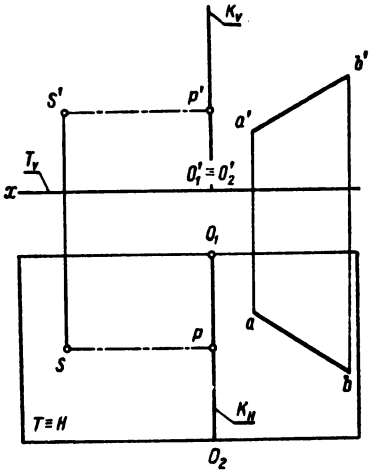

Fig. 382

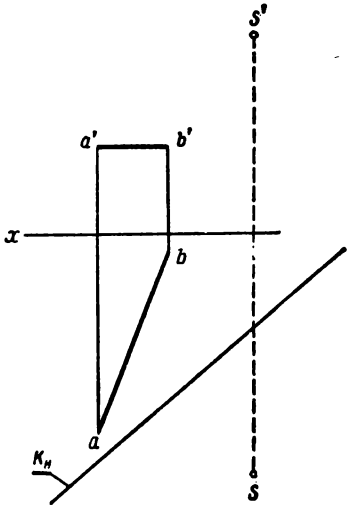

Fig. 383

Exercises

1. Given the orthogonal projections of a system of planes of linear perspective and line-segment AB (Fig. 382). To construct the perspective and secondary projection of this segment. To determine two singular points of the straight line (the beginning and the infinite point).

2. To solve the problem for a horizontal segment AB (Fig. 383).

3. Through a given point E to draw a straight line parallel to line AB (Fig. 384).

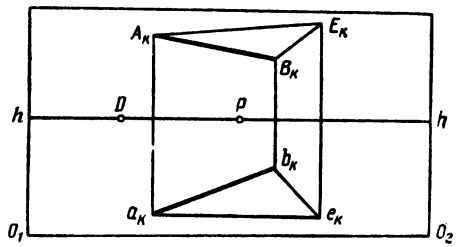

Fig. 384

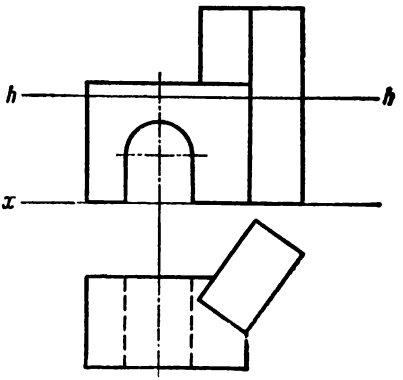

Fig. 385

4. To determine the perspective of the centre of gravity of triangle ABE (Fig. 384).

5. To construct by the architectural method the perspective of a group of geometric solids for a given horizon line (Fig. 385).

6. Given the perspective and secondary projection of one of the diagonals of a rectangular parallelepiped, the base of which is situated in the ground plane and the front face is parallel to the picture plane. To construct the perspective of the parallelepiped and determine the true lengths of its edges and diagonal.

7. In the ground plane to draw a straight line through point E to make an angle of 60° with line AB, which also lies in the ground plane.

CHAPTER XIV

GEOMETRIC TRANSFORMATIONS INVOLVING CENTRAL AND PARALLEL PROJECTION

Sec. 73. Fundamentals

In geometry, transformations may be specified in different ways. The ones most often considered are point transformations: to each point of one figure there corresponds a definite point of another figure. In point transformation, each figure (the inverse image) considered as a set of points is transformed into a new figure, called the image of the original one.

Points corresponding to one another are called *corresponding* points. Lines and segments defined by corresponding points are also termed corresponding.

A construction in which a new figure is obtained from a given figure is called transformation of a geometric figure

Instances of such transformations are axial and central symmetry and homothetic transformation, which are studied in plane geometry.

Each geometric transformation of a figure is associated with a certain distortion of the shape of the figure and the dimensions of the separate parts. At the same time, the specific properties of the figures remain unchanged during such a transformation.

Properties (of figures) that remain unchanged in a transformation are said to be invariant under that transformation. To illustrate, the transformation of a plane figure in parallel projection from one plane to another has the following invariants: (1) linearity of the segments, (2) parallelism of the straight lines, (3) the ratio of lengths of segments of one and the same straight line, and (4) the ratio of segments of two parallel straight lines (the proof is given in Sec. 3)

These invariants are characteristic not only of parallel projection but also of certain other geometric transformations, such as homothetic transformations.

All transformations that possess the foregoing invariants are known as affine transformations. Parallel projection is a special case of such transformations.

The correspondence set up between figures by means of parallel projection is called a perspective-affine correspondence.

Central projection with nonparallelism of the planes of the image and inverse image does not yield an affine transformation but a more general projective transformation.

In order to establish a one-to-one point correspondence between two planes in the case of central projection, the space and plane of Euclidean geometry are supplemented by infinite elements: projective space and a projective plane.

Sec. 74. The Projective Plane and Projective Space

The plane and space of elementary geometry are Euclidean, after the Greek geometer Euclid. Euclidean space does not ensure complete one-to-one point correspondence between two planes in the case of central projection (perspective)

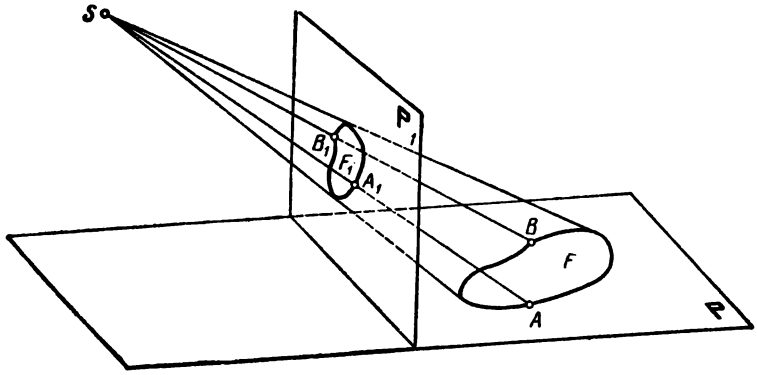

Fig. 386

Fig. 386 shows the projection of figure F_1, located in the plane P_1, from point S (the centre of projection) to figure F in plane P.

A one-to-one correspondence is set up between the points of figures F_1 and F To point A_1 of figure F_1 there will correspond a point A of figure F. Conversely, to each point B of figure F there will correspond a point B_1 of figure F_1.

However, it is not possible to extend this assertion to all points of planes P and P_1 in Euclidean space.

Indeed, consider on plane P_1 a point K_1 located on ray SK_1 parallel to plane P (Fig. 387). This ray has no point of intersection with plane P. Consequently, there is no point on plane P that corresponds to point K_1 in plane P_1.

Now draw through the centre of projection S a ray (parallel to plane P_1) that will intersect plane P in point L. For this point L of plane P it will be impossible to define a corresponding point (inverse image) on plane P_1.

This circumstance necessitates supplementing the Euclidean plane with new elements called ideal elements or elements at infinity. Such an addition leads us to the formation of a new geometric entity called the projective plane.

In this process, each straight line is supplemented by a point at infinity: parallel lines have a common point at infinity, nonparallel lines, different points at infinity. All points at infinity belong to a straight line at infinity.

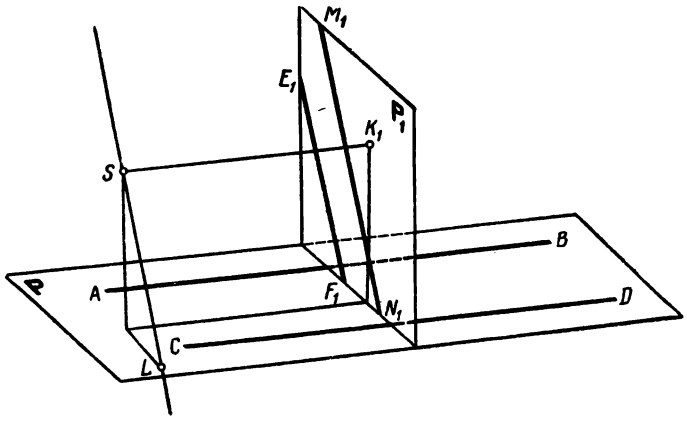

Fig. 387

In contrast to ideal points (points at infinity), the points of the Euclidean plane are called proper points. The straight lines of the Euclidean plane supplemented by ideal points are known as proper straight lines.

To summarize, then, a Euclidean plane supplemented by a line at infinity is called a projective plane.

The following statements hold in the projective plane: (1) only one straight line passes through any two distinct points; (2) any two distinct straight lines have only one common point. Analogously, projective space is obtained from Euclidean space by the addition of ideal points (points at infinity), ideal straight lines and an ideal plane. In projective space, like in a plane, any two straight lines lying in a single plane will always intersect. In projective space, any two planes intersect along a straight line. Finally, any straight line not lying in a plane will always intersect the latter.

By augmenting Euclidean space to projective space we obtain a correspondence between planes P_1 and P (see Fig. 387) in central projection that becomes one-to-one: if ray SK_1 is parallel to plane P, then to point K_1 there corresponds a point at infinity of plane P,

which is added to the straight lines AB and CD of plane P (these lines are parallel to ray SK_1). The ideal point of straight lines E_1F_1 and M_1N_1 drawn in plane P_1 parallel to ray SL is the inverse image of point L in plane P.

Having set up a space in which projection may be executed without any exceptions, we may now pass to a study of the correspondence of two plane figures which is obtained as a result of central projection.

Sec. 75. Perspective Collineation and Homology

Let us consider the correspondence of plane figures in the case of central projection. Let there be given two planes P and P_1 that intersect along the straight line O_1O_2 (Fig.388), and let a certain projection

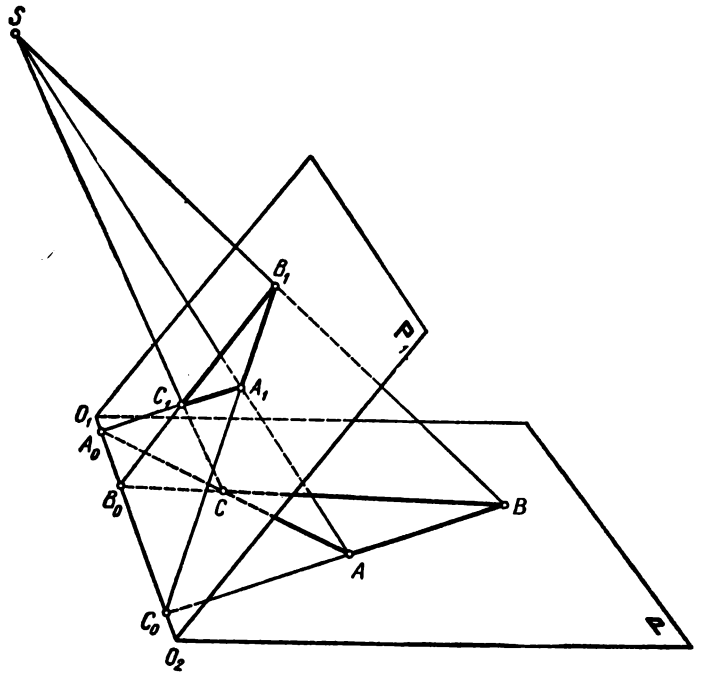

Fig. 388

be established from the centre of projection S in which point A_1 of plane P_1 is projected into point A of plane P. Points A_1 and A are corresponding points. Each line of plane P_1 will project into a line of plane P. For instance, to line A_1C_1 of plane P_1 there corresponds line AC of plane P and conversely.

In the case of central projection, to points lying on a straight line of plane P_1 there correspond points lying on the corresponding line of plane P. Consequently, in central projection we have a one-to-one transformation in which points lying on a straight line of plane P_1 go into rectilinearly located points of plane P.

The correspondence that possesses this property is known as collineation.

Collineation in which corresponding points lie on projecting rays that intersect in the centre of projection S is termed perspective collineation.

The locus of singular points of this collineation will be the straight line O_1O_2 along which planes P and P_1 intersect. Each point of this straight line will correspond to itself, since each of the points of line O_1O_2 coincides with its central projection. These points are known as dual points. The line O_1O_2 which coincides with its projection is also dual. This line is the axis of perspective collineation.

Two corresponding straight lines of planes P_1 and P intersect on the axis of the collineation. Indeed, let us consider the corresponding lines AB and A_1B_1 lying in the same projecting plane SAB (see Fig. 388). Here, lines AB and A_1B_1 lie in a single plane and intersect in point C_0. This point lies in three planes: plane P (as a point of line AB), plane P_1 (as a point of line A_1B_1) and in the projecting plane that contains lines AB and A_1B_1. Hence, point C_0 must be on line O_1O_2 along which planes P and P_1 intersect.

Repeating similar reasoning for each pair of corresponding sides of the triangles $A_1B_1C_1$ and ABC (see Fig. 388), we conclude that if in two triangles the straight lines connecting corresponding vertices pass through a single point S, then the three intersection points $A_0B_0C_0$ of the corresponding sides lie on a straight line. This is a statement of the theorem of Desargues which also holds for triangles lying in one plane.

It may be demonstrated that the perspective-collinear correspondence of the points of two planes is not upset if one of the planes is rotated about the axis of the collineation.

In other words, when plane P_1 is rotated, all straight lines AA_1, BB_1, ... (see Fig. 388) connecting corresponding points remain straight lines intersecting in one point, and this point—the centre of projection—does not remain stationary.

Now let there be a perspective-collinear correspondence of points of two planes P_1 and P, in which to a segment A_1B_1 of P_1 there corresponds a segment AB of P. Point S (Fig. 389) is the centre of projection. After rotation of plane P_1 through an arbitrary angle φ about the axis of the collineation O_1O_2, we establish the relative positions of the projecting rays AA_1 and BB_1. P_2 marks the new position of the plane. After rotation, A_1B_1 will take up the position A_2B_2,

but will, as before, intersect AB in the fixed point C_0 lying on the axis of rotation O_1O_2. Hence, the straight lines AB and A_2B_2 lie in a single plane. Lines AA_2 and BB_2 lie in the same plane, which means that these lines intersect in a certain point S_1. The foregoing reasoning holds for any position of a rotating plane and for any pair of any three straight lines AA_1, BB_1, CC_1 not lying in one plane but connecting corresponding points But if of three straight lines not lying in one plane, every two intersect, then all these lines have a

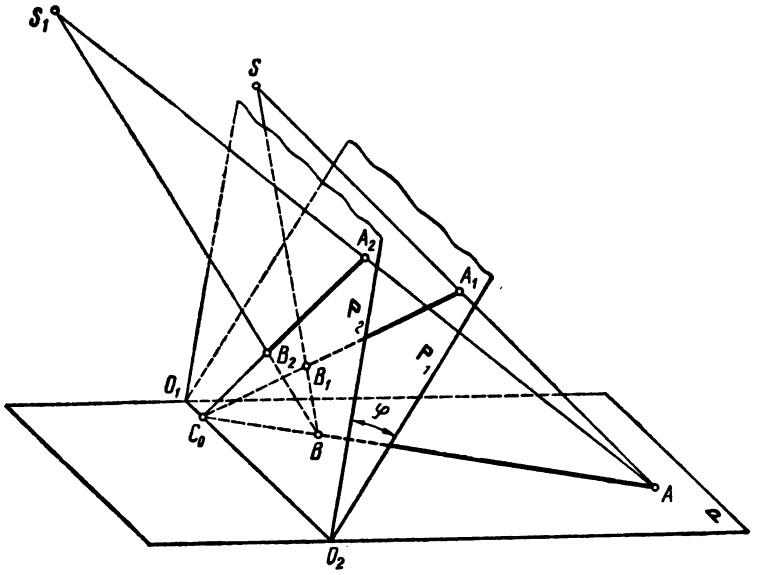

Fig. 389

single point of intersection. Adding to these three lines any other one connecting corresponding points, we conclude that all straight lines connecting corresponding points intersect in one point upon rotation of plane P.

In the process of rotation, both planes may be brought to coincidence. Correspondence between the points of such coincident planes may no longer be regarded as the result of central projection. It would then be a case of the transformation of the points of one plane into other points of the same plane in which we retain the rectilinear arrangement of the points, and all points of a certain straight line remain fixed. Such a one-to-one transformation of a projective plane into itself is called a homology.

All straight lines joining corresponding points pass through one point S_0, the centre of homology (Fig. 390). The straight line contain-

ing points that transform into themselves (dual points) is called the axis of homology (line O_1O_2).

Pairs of corresponding straight lines intersect on the axis of homology. Figures that correspond to one another are in this case called homologous figures. A homology is specified by giving the centre S_0, the axis O_1O_2 and pairs of corresponding points A and A_1 situated on the same straight line as S_0 (see Fig. 390). In this case, for

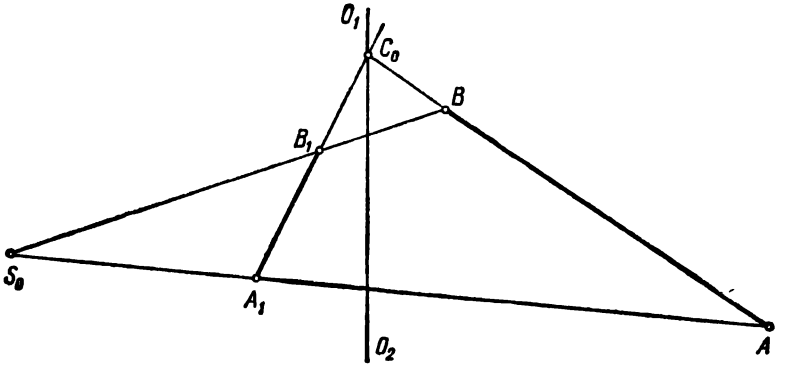

Fig. 390

each point of the plane (say B) we can define the corresponding point B_1. For this purpose we do as follows. The straight line A_1B_1 that corresponds to AB must pass through the dual point C_0 of the latter. This enables us to construct C_0A_1 in Fig. 390. The desired point B_1 must lie on this straight line C_0A_1 and on the projecting line S_0B. The intersection of these lines yields point B_1.

Fig. 391 illustrates a case when one of the given corresponding points is an infinite point of S_0A; in other words, in this homology, to the point A there corresponds an infinite point of the straight line S_0A. To construct point B_1, which corresponds to point B, we extend AB (as before) to intersection with the axis O_1O_2 in point C_0. C_0B_1, corresponding to AB, will pass through this point. C_0B_1 must intersect S_0A at infinity, which means that C_0B_1 will be parallel to S_0A. The desired point B_1 must belong to C_0B_1. It is defined as the point of intersection of the straight line C_0B_1 with the projecting line S_0B.

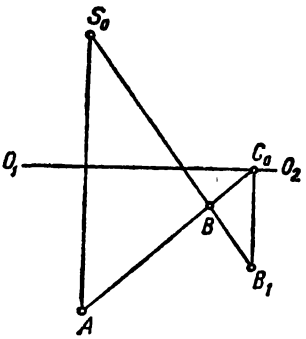

Fig. 391

Homology plays an essential role in the theory of linear perspective. Here,

a correspondence is established between the perspective of a figure and the figure itself; the plane of the latter must be coincident with the picture plane K.

Coincidence is achieved by rotating the figure together with its plane about a straight line in which the plane of the figure intersects the picture plane. This line serves as the axis of homology because its points are transformed into themselves.

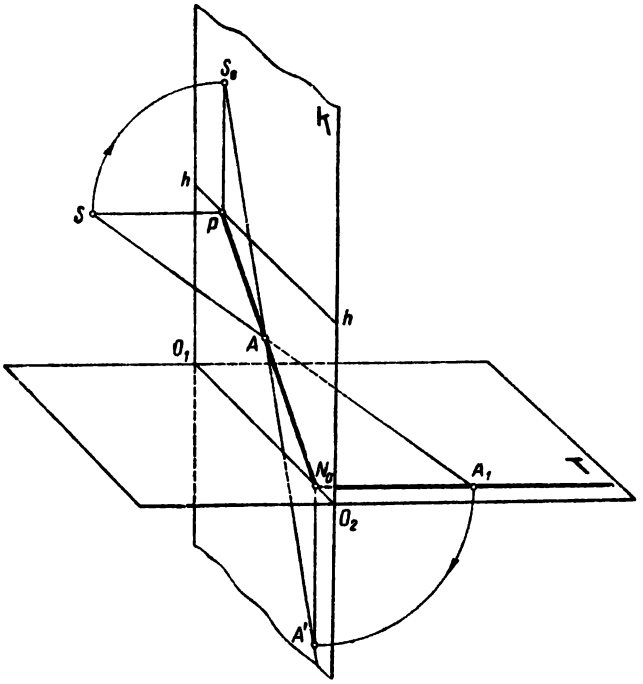

Fig. 392

Now the centre of homology will be the point of the picture with which the centre of projection (point of sight S) is coincident. Projective geometry gives proof that when the plane of a given figure is in rotation, the centre of projection rotates in a circle, the plane of which is perpendicular to the axis of homology. The centre of this circle is the point of intersection of the picture plane and a straight line passing through the point of sight S perpendicular to the axis of homology and parallel to the plane of the figure.

Without giving proof of this statement for the general case, we shall show how to determine the position of the centre of homology

S_0 when bringing into coincidence with the picture plane figures lying in the ground plane T. We refer to Fig. 392. Point A_1 lies in the ground plane T. Through this point we draw a straight line A_1N_0 perpendicular to the picture plane K

In the collinear correspondence established between the points of planes T and K in a projection from S, straight line N_0P of plane K corresponds to A_1N_0 of plane T. It will be noted that to the point P (the principal point) there corresponds a point at infinity on the line A_1N_0. Using the projecting ray SA_1, we define the perspective A of point A_1. When plane T is brought into coincidence with the picture plane and when the ground line of the picture O_1O_2 is taken as the axis of revolution, point A_1 will take up the position labelled A' ($A'N_0 = A_1N_0$). The projecting ray will have to pass through the corresponding points A' and A and the centre of homology. Also, the required centre will have to belong to the straight line joining point P and the infinite point of line $A'N_0$ that is homologous to it. This straight line PS_0 passes through the principal point P parallel to $A'N_0$. The intersection of $A'A$ and PS_0 yields the position of the centre of homology: point S_0.

It may readily be shown that segment PS_0 is equal to the principal distance SP. Indeed, from the similarity of the triangles APS and AN_0A_1 it follows that

$$\frac{SP}{PA} = \frac{A_1N_0}{N_0A} \tag{1}$$

whence

$$SP = PA\,\frac{A_1N_0}{N_0A} \tag{2}$$

But triangles APS_0 and AN_0A' are likewise similar, and therefore

$$\frac{S_0P}{PA} = \frac{A'N_0}{N_0A} \tag{3}$$

and

$$S_0P = PA\,\frac{A'N_0}{N_0A} \tag{4}$$

Comparing equations (2) and (4), we conclude that

$$S_0P = SP$$

Thus, the centre of homology is displaced vertically from the principal point P a distance equal to SP (the principal distance).

Having established the position of the centre of homology and the correspondence of the principal point P to a point at infinity of straight lines of the ground plane, each of which is perpendicular to

the ground line, let us now take up the solution of certain problems in linear perspective.

Problem 1. To determine the true length of a line-segment lying in the ground plane if its perspective A_KB_K is given (Fig. 393).

To determine the length of the segment, rotate the ground plane T about the ground line of the picture to coincidence with the picture. The centre of perspective, as has just been demonstrated, coincides with the picture at point S_0, which is the centre

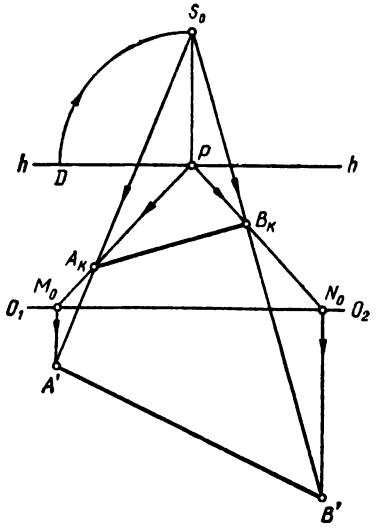

Fig. 393

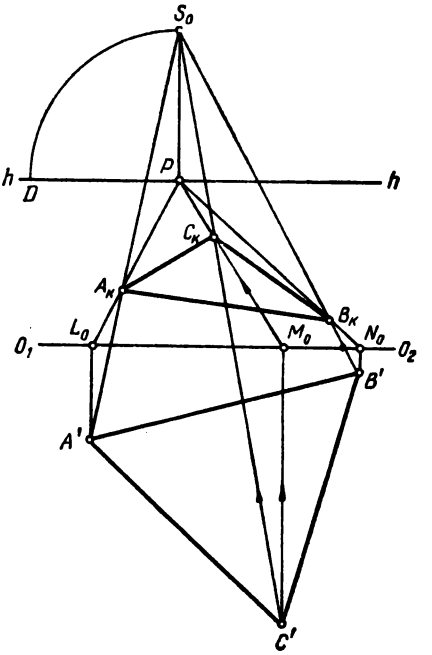

Fig. 394

of homology linking the perspective of the given segment with its coincident image $A'B'$. The ground line of the picture O_1O_2 serves as the axis of this homology. We stress the fact that the infinite point of line S_0P corresponds to the point P.

Taking advantage of these data, for A_KB_K we can construct a corresponding line-segment $A'B'$. The latter will define the required length of the line-segment given by the perspective A_KB_K.

Point A' (homologous to A_K) is found as the point of intersection of the straight lines S_0A_K and M_0A'. The latter is a line that corresponds to PM_0. It will be noted that to point P of line M_0P there corresponds a point at infinity of line M_0A'. Point B' is found in the same way.

Similarly we can determine the true length of the secondary projection of any line-segment.

Problem 2. To construct the perspective of a regular triangle located in the ground plane, given the perspective $A_K B_K$ of one of its sides (Fig. 394).

This problem is obviously the inverse of the preceding one. Its solution requires finding, by the foregoing method, the coincidence of the given side $A'B'$. On this segment we construct a regular triangle $A'B'C'$ and then determine the triangle $A_K B_K C_K$ homologous to the coincident one.

Sec. 76. Perspective-affine Correspondence

Correspondence that is established between figures by means of parallel projection is called perspective-affine correspondence.

Perspective-affine correspondence is a special case of perspective collineation when the centre of projection is a point at infinity.

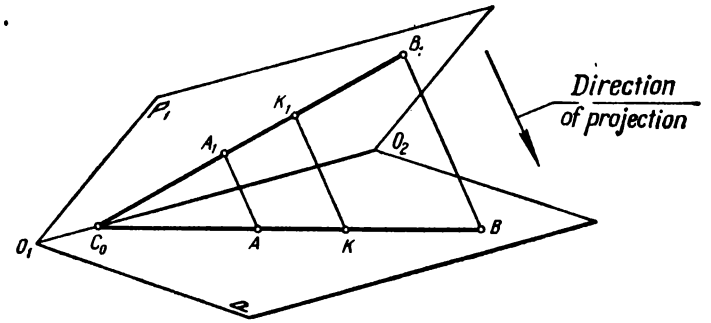

Fig. 395

Affine correspondence possesses the general properties of perspective collineation and the specific properties of parallel projection.

In affine correspondence as in the case of any perspective collineation:

(1) to every point A of one plane there corresponds an affine point A_1 of the second plane (Fig. 395);

(2) to every straight line AB of one plane there corresponds an affine straight line A_1B_1 of the second plane;

(3) affine points are located on straight lines passing through the centre of projection (for a centre at infinity, the projecting rays will be parallel);

(4) affine lines AB and A_1B_1 intersect on the affine axis, which is the intersection line O_1O_2 of the given planes P and P_1. This axis is the locus of dual points;

(5) the properties of points belonging to lines are preserved; that is, if point K lies on AB, the affine point K_1 belongs to the affine line A_1B_1.

In addition, affine correspondence has the following properties due to the specific position of the centre of projection: (1) to the parallel straight lines of one plane there correspond parallel straight lines

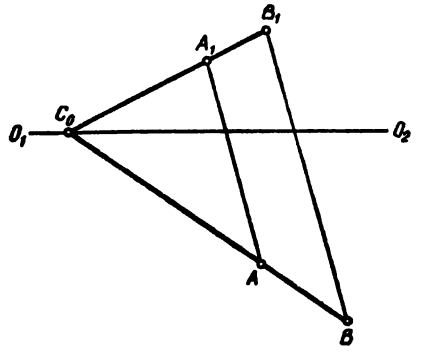

Fig. 396

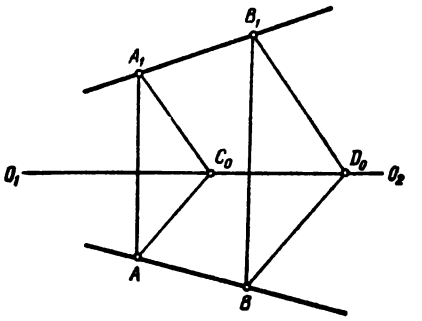

Fig. 397

of the second plane; (2) the ratio of lengths of line-segments of one line or of parallel lines is equal to the ratio of affine line-segments (see Sec. 3 for the proof). Descriptive geometry makes wide use of these properties for depicting solids and plane figures on projection drawings.

The affine correspondence established between two planes in parallel projection is preserved when the planes are brought into coincidence.

But here the straight lines connecting affine points will determine not the direction of projection but the affine direction.

The affine correspondence of two coincident planes may be specified by the affine axis O_1O_2 and two corresponding points A and A_1 (Fig. 396). To find point B_1 (the affine point of an arbitrary point B), draw the straight line AB and mark the point C_0 of its intersection with the affine axis. Points C_0 and A_1 define a straight line which is affine to AB. The straight line A_1C_0 must contain the required point B_1, which is found by drawing BB_1 parallel to AA_1 (parallel to the affine direction).

When AB intersects the affine axis outside the drawing (Fig. 397), the problem may have a different solution. Through point A draw

an arbitrary straight line AC_0 which intersects the affine axis within the drawing; then construct a straight line C_0A_1 which is affine to this one. Then through point B draw a straight line BD_0 parallel to AC_0, and D_0B_1 (corresponding to AC_0) parallel to C_0A_1 (parallel lines correspond to parallel lines). The straight line that passes through B parallel to the affine direction will intersect D_0B_1 in the desired point.

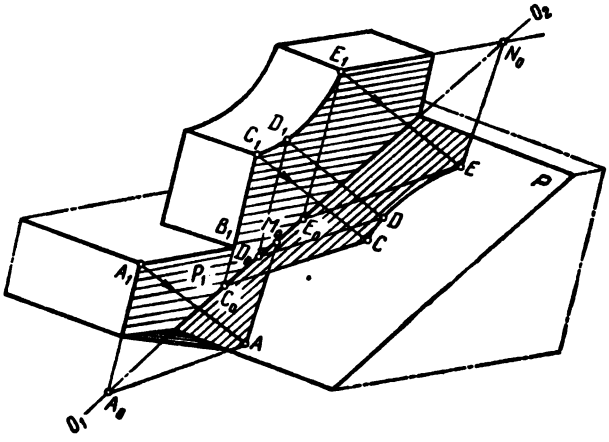

Fig. 398

This construction is employed when it is required to draw a straight line A_1B_1 through an inaccessible vanishing point of two other straight lines (through the point of intersection of AB and O_1O_2).

An instance of affine figures is a given figure and its shadow on another plane. Fig. 398 shows the construction of the shadow of a figure (lying in plane P_1) on the inclined plane P. The shadow of point A_1 on plane P was given (point A). Having two corresponding points A_1 and A and the affine axis (line O_1O_2), it is easy to find the points C, D, E, which are affine points of C_1, D_1, and E_1. In determining these points we repeated the constructions shown in Fig. 397. Thus, point C (the affine point of C_1) is found by means of lines C_1C_0 and C_0C, respectively parallel to A_1A_0 and A_0A. The shadows of the remaining points are constructed in similar fashion.

Another instance of affine figures is the perspective of a figure and its secondary projection (Fig. 399).

Indeed, in this correspondence the affine points of the figures are perspectives A_K, B_K, C_K and the secondary projections a_K, b_K, c_K of the same points of space. But we know that the perspective of a point

and its secondary projection are always located on a single perpendicular to the ground line. Hence, straight lines joining each pair of affine points are parallel (that is, they intersect at infinity). Applying the theorem of Desargues for a plane, we conclude that the three intersection points L_0, M_0, N_0 of three pairs of affine sides of the triangles $A_KB_KC_K$ and $a_Kb_Kc_K$ lie on a single straight line O_1O_2 (the affine axis). This straight line is the perspective of the intersection line of the ground plane and the plane containing the triangle.

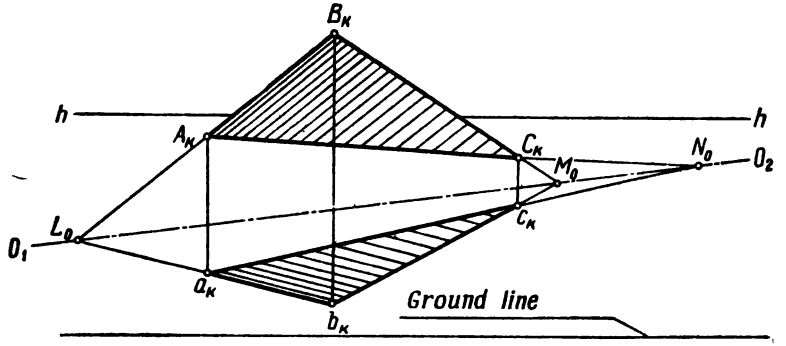

Fig. 399

Analogously, we can show that the orthogonal projections of a plane figure are also affine figures.

In this case, the affine direction will be perpendicular to the x-axis. The affine axis will pass through the intersection points of the affine lines. In Fig. 400 these points are M and N. The former is defined by the intersection of ab and $a'b'$, the latter is the point of intersection of ac and $a'c'$.

Coincidence of the horizontal projection m and the vertical projection m' indicates that point M is equally distant from planes H and V and lies in the second quadrant. Also at equal distances from the projection planes is the point N, but in the given case it lies in the fourth quadrant. Hence, points M and N determine the intersection line of the plane of the triangle and the bisector plane of the second and fourth quadrants of space. This is the line that serves as the affine axis.

Thus, in the orthographic representation we establish a definite affine correspondence between the vertical and horizontal projections of points of an oblique plane.

Figures lying in the projecting planes will not obey this correspondence because one of the projections of the figures degenerates into a straight line.

The laws of affine correspondence permit us to solve a variety of problems of orthogonal projections.

Fig. 401 is a case for determining the points of intersection of a straight line EF with a plane given by two parallel lines AB and CD ($e'f'$ is parallel to $a'b'$).

First find the affine axis (line O_1O_2) as the line of intersection of the plane of two parallel straight lines and the bisector plane of the second quadrant. The problem now consists in constructing a pair of points that are affine in the given correspondence and lie on the

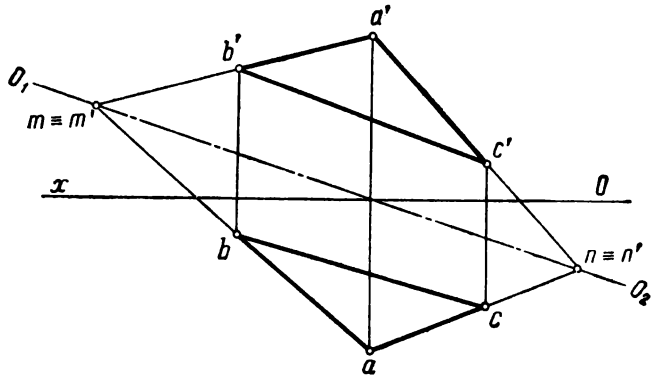

Fig. 400

straight lines ef and $e'f'$. We note that ef and $e'f'$ are not affine in the given correspondence.

For $e'f'$ we construct its affine line le_1. Line le_1 is drawn parallel to ab, since parallel lines ab and le_1 must correspond to parallel lines $a'b'$ and $l'e'$.

The intersection point K of lines le_1 and ef is one of the required points. Its affine point K' is found directly with the aid of the affine direction. The points thus found are the orthogonal projections of the point of intersection of the line and plane.

The use of projecting planes in this case (taking into account the limited size of the drawing) would have appreciably complicated the solution.

The foregoing method may be used to solve problems relating to the intersection of polyhedrons and ruled surfaces by a plane; and when determining the figure which is affine to the given one, we repeatedly execute the geometric constructions shown in Figs. 396 and 397, where for a given point B we found the affine point B_1.

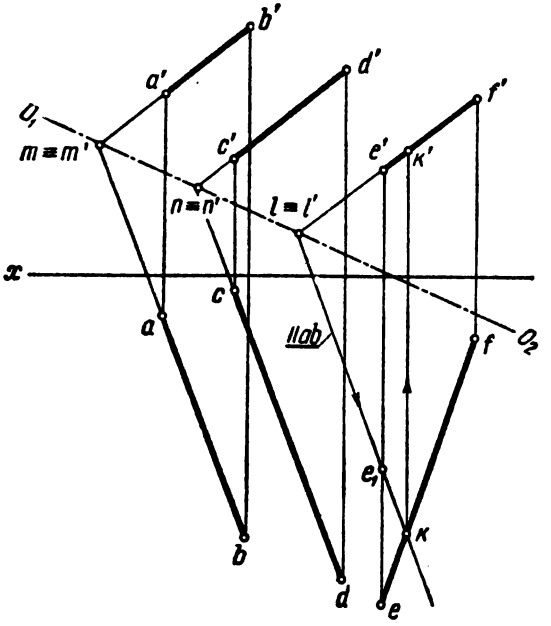

Fig. 401

Sec. 77. A Curve Affine to a Circle

A curve that is affine (affine-corresponding) to a circle is called an ellipse.

We pose the following problem: to construct an ellipse that is affine to a given circle. Let there be given the affine axis O_1O_2, two affine points D_1 and D and a circle with centre C_1 (Fig. 402).

First find point C which is affine to the centre of the circle. Draw through C_1 an arbitrary straight line C_1N_1 and construct a straight line corresponding to it. To do this, draw D_1L_1 parallel to C_1N_1, then join points L and D and, finally, construct NC parallel to LD. Point C is determined by the intersection of NC and C_1C, the latter being parallel to the affine direction D_1D.

We know that the centre of a circle is its centre of symmetry. All diameters of a circle are cut in half in the centre. This property is preserved under an affine transformation, in other words, the ellipse is also a central-symmetric curve. Each diameter of the ellipse* is divided in half in its centre C.

* The diameters of an ellipse are chords that pass through its centre.

In Fig. 402 the diameter AB of the ellipse is constructed with the aid of lines A_1A and B_1B, which are parallel to the affine direction.

Now, in the given circle draw diameter E_1F_1 perpendicular to diameter A_1B_1 and construct its affine diameter EF of the ellipse. To the mutually perpendicular diameters of the circle there correspond, generally speaking, the nonperpendicular diameters of the ellipse. But both possess the common property of conjugacy.

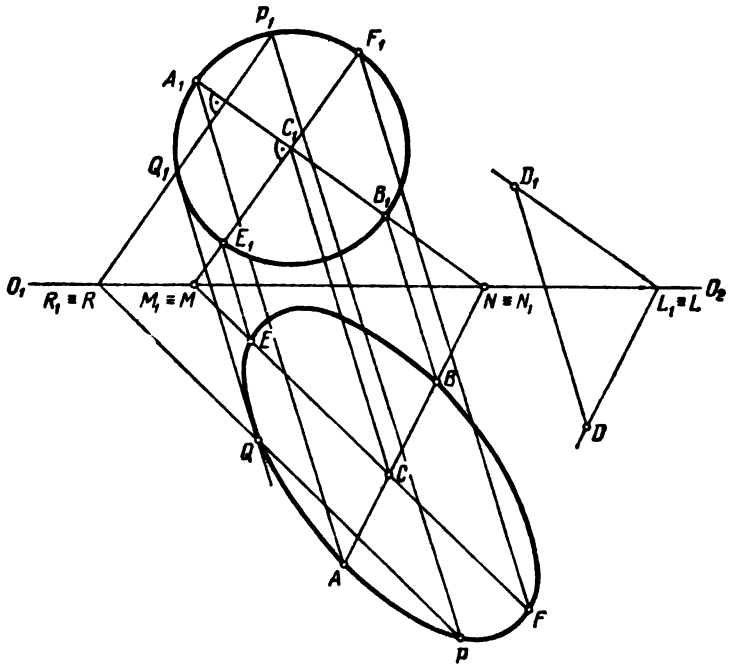

Fig. 402

Two diameters are conjugate if each of them divides in half the chords parallel to the other diameter.

By invoking familiar theorems of plane geometry we can readily prove that the mutually perpendicular diameters of a circle are also conjugate diameters.

We shall demonstrate that to conjugate diameters of a circle there correspond conjugate diameters of an ellipse.

In a circle draw one of the chords P_1Q_1 parallel to the diameter E_1F_1 and construct a chord PQ affine to it which will be parallel to the diameter EF of the ellipse. The diameter A_1B_1 divides in half the chord P_1Q_1 perpendicular to it; for this reason the chord PQ must

be divided in half by the diameter AB of the ellipse. This assertion may be extended to other chords that are parallel to the diameter E_1F_1.

Thus, the diameter AB is a locus of midpoints of chords that are parallel to the diameter EF. Analogously, it may be demonstrated that diameter EF likewise divides in half chords parallel to the diameter AB.

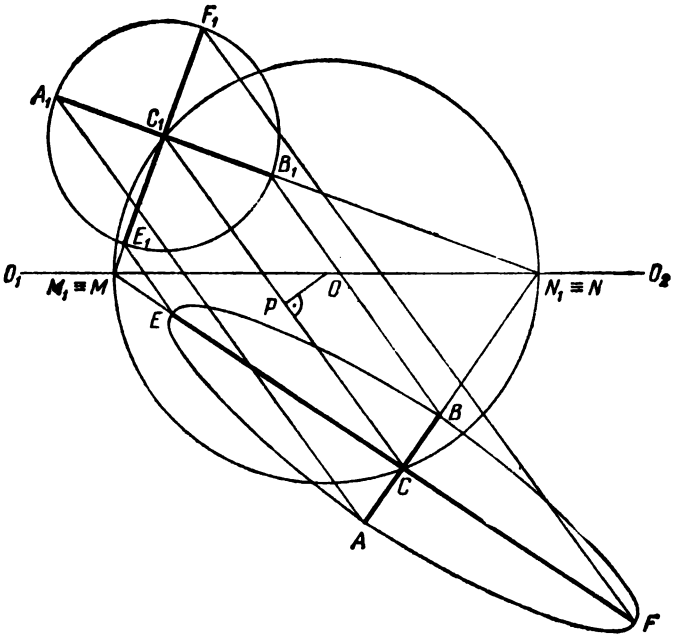

Fig. 403

Consequently, to perpendicular diameters of a circle there correspond conjugate diameters of an ellipse. It is proved in projective geometry that for any affine transformation of a plane there exist two such mutually perpendicular, so-called principal, directions which again transform into mutually perpendicular ones.

Determining the principal directions involves solving the following problem. Given: the affine axis O_1O_2 and affine points C_1 and C (Fig. 403). Required: to draw through C_1 two mutually perpendicular straight lines so that the straight lines that are affine to them should also be orthogonal In other words, it is required to construct a quadrangle C_1MCN, the angles at points C_1 and C of which must be

right angles, and the vertices M and N must lie on the straight line O_1O_2.

We know that it is possible to circumscribe a circle about a quadrangle, the sum of the opposite angles of which is 180°. The right angles at vertices C_1 and C will rest on a common diameter MN. Consequently, the centre of the circumscribed circle must lie on the affine axis.

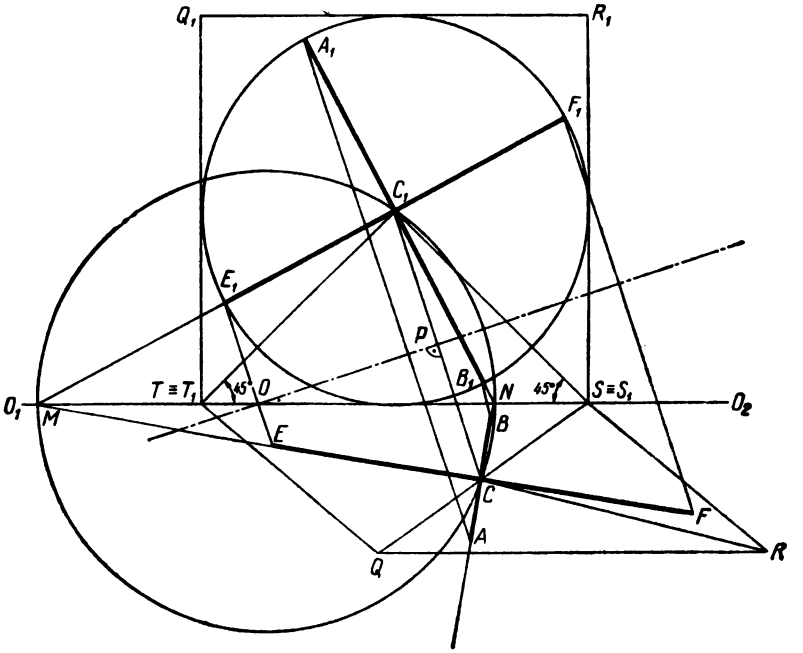

Fig. 404

The position of this centre O is determined by the intersection of the affine axis and the perpendicular PO erected to the straight line C_1C at the midpoint. Drawing a circle of radius OC, we find points M and N in which it intersects the affine axis. Joining M and N with C_1, we get the desired straight lines. Indeed, lines MC_1 and NC_1 are perpendicular since the angle they make rests on the diameter. For that same reason, MC and NC are perpendicular.

And so if the conjugate diameters of a circle coincide with the principal directions, to them will correspond the conjugate and orthogonal diameters of an ellipse.

Such diameters of an ellipse are called its axes. The construction of the axes AB and EF of an ellipse is given in Fig. 403.

In conclusion we consider determining the axes of an ellipse that must be inscribed in a given parallelogram $QRST$ (Fig. 404).

As a first step, construct a square $Q_1R_1S_1T_1$ that is affine to the given parallelogram.

To simplify the drawing, bring the affine axis O_1O_2 into coincidence with the side ST. Drawing diagonals in the parallelogram and the square, we find the affine points C and C_1. It remains to construct a circle that passes through these points and has a centre on the affine axis O_1O_2; that is to say, the subsequent graphical constructions are essentially the same as the constructions involved in determining the principal directions.

The dimensions in Fig. 404 may be reduced somewhat if in place of a square related to the given parallelogram we draw a triangle TC_1S.

The foregoing method is recommended for use in constructing the axonometric projections of a circle.

This chapter has dealt with certain propositions of projective geometry. More detailed information about geometric transformations and the laws of perspective collineation may be found in the special literature.

CHAPTER XV

PROJECTIONS WITH ELEVATIONS

Sec. 78. Fundamentals of the Method. Projections of Points

The method of projections with elevations is widely used in designing railways and highways, power stations, canals, aerodromes, and construction sites; in other words, in cases when the height of the object is not great relative to its length and width. This method is also used to depict surfaces that do not obey any geometric law, such as the earth's surface.

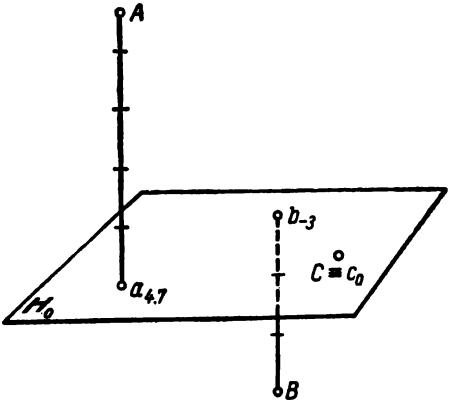

Fig. 405

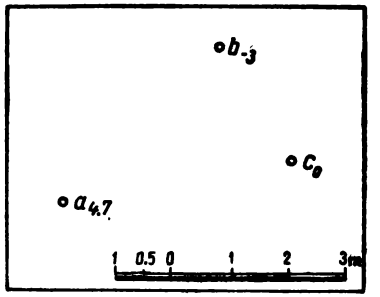

Fig. 406

Essentially, the method of projections consists in the fact that a given object is orthogonally projected onto only the horizontal plane H_0, zero-elevation plane. To obtain an exact image of the object, the projections of individual points are labelled with numbers (written to the right of the projection) indicating the distances (usually in metres) from the given points to the H_0 plane. These numbers are called elevations. A minus sign is prefixed to the elevation if the point is located below the zero-level plane.

Fig. 405 gives a pictorial impression of how the projections of points are constructed. The numbers used to label points A, B and C indicate that point A is located above the H_0 plane a height of 4.7 metres, point B is located below the H_0 plane 3 metres from the zero level, and point C lies in the zero-level plane.

Fig. 406 shows these three points depicted in projections with elevations. Let us agree to call such drawings plans. Plans must also include a linear scale for use in solving metrical problems.

Sec. 79. The Straight Line

We introduce certain concepts that will be essential to what follows.

Let us agree to call the length of the horizontal projection of a line-segment the horizontal equivalent of the line. For this line-

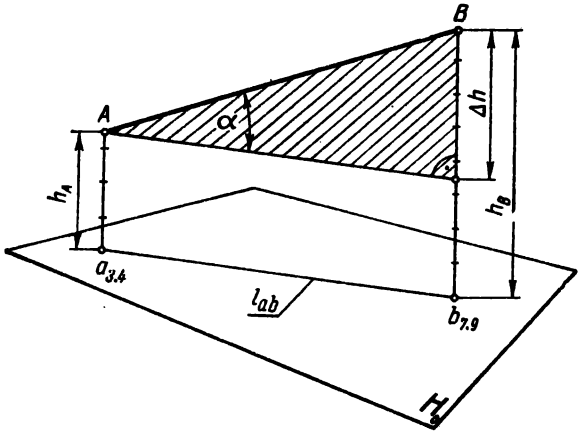

Fig. 407

segment (Fig. 407) we form a ratio of height differences (difference of elevation) $h_B - h_A$ to the horizontal equivalent l_{ab} and denote this fraction by i:

$$i = \frac{h_B - h_A}{l_{ab}}$$

The quantity $i = \tan \alpha$ is called the slope of the line, and α, the angle of inclination of the line to the H_0 plane.

There is a definite geometrical meaning in the inverse fraction $\frac{l_{ab}}{h_B - h_A}$, which indicates the magnitude of the horizontal equivalent corresponding to a unit of elevation of point B above A.

The horizontal equivalent of a straight line corresponding to a unit of elevation is called the interval of the line and is designated by L. This we have $l_{ab} = L$ for $h_B - h_A = 1$.

From the foregoing definitions it follows directly that the slope and the interval of a straight line are reciprocal quantities: $i = \frac{1}{L}$.

The interval of line is used in graduating the projection of a straight line. To graduate the projection of a line means to determine points on it with a difference of elevations equal to unity.

To graduate the projection of segment AB (Fig. 408), it is first necessary to determine the interval $L = \frac{l_{ab}}{h_B - h_A}$. In our example, $l_{ab} = 12.2$ metres, $h_B - h_A = 4.5$ m and $L = 2.7$ m.

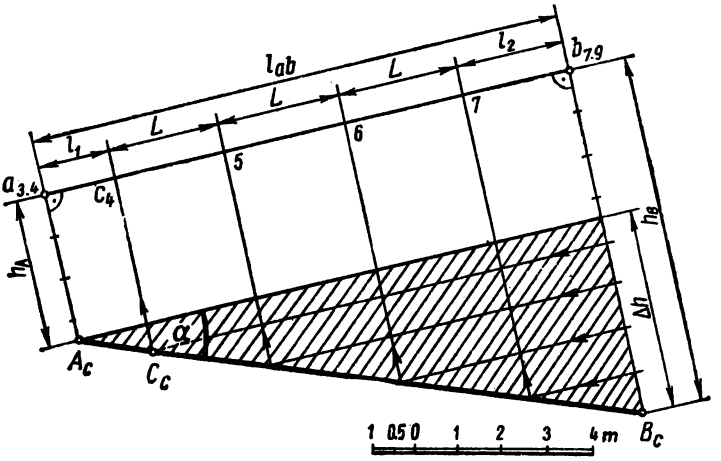

Fig. 408

From point $a_{3.4}$ to the nearest larger point with an integral number, we have to lay off a segment l_1 defined from the following proportion:

$$\frac{l_1}{L} = \frac{4 - 3.4}{1} \quad \text{whence} \quad l_1 = 0.6 \times 2.7 = 1.62 \text{ m}$$

The subsequent points with elevations 5, 6 and 7 will be separated by the distance $L = 2.7$ m. If the projection of the segment is continued leftwards from point $a_{3.4}$ (by laying off intervals L from c_4), we get points with elevations 3, 2, 1, and, finally, 0, which is the horizontal trace of the straight line.

In Fig. 408 we find the true length of AB; to do this, we bring the trapezoid $ABa_{3.4}b_{7.9}$ (see Fig. 407) into coincidence with the plane of the drawing; the projecting rays $Aa_{3.4}$ and $Bb_{7.9}$ serve as the bases, segment AB and its projection $a_{3.4}\ b_{7.9}$ as the nonparallel sides. The projection $a_{3.4}\,b_{7.9}$ is the axis of rotation. In the coincident position, the trapezoid took up the position $A_c a_{3.4} b_{7.9} B_c$. Here, points A_c and B_c are found to lie on perpendiculars to the axis of rotation,

the former at a distance $h_A = 3.4$ m from the projection $a_{3.4}b_{7.9}$, the latter at a distance $h_B = 7.9$ m. The two segments representing the parallel sides of the trapezoid are constructed to the scale of the drawing. For the sake of simplicity it is common to draw only a right triangle (shown cross-hatched in Figs. 407 and 408) instead of the whole trapezoid; the sides of the triangle are the projection of the given segment and Δh (the elevation of point B above A). The hypotenuse of the triangle determines the required length of the segment AB, while the angle between A_cB_c and the horizontal side (angle α) is the angle between the segment of AB and the H_0 plane; α is the inclination of this line to the H_0 plane.

Fig. 408 also depicts a graphical graduation of the projection of segment $a_{3.4}b_{7.9}$ by means of two pencils of parallel straight lines indicated by arrows.

Sec. 80. The Relative Positions of Two Straight Lines

Let there be given the intersecting projections of two straight lines AB and CD (Fig. 409). To evaluate the relative positions in space of the lines themselves, we have to graduate their projections

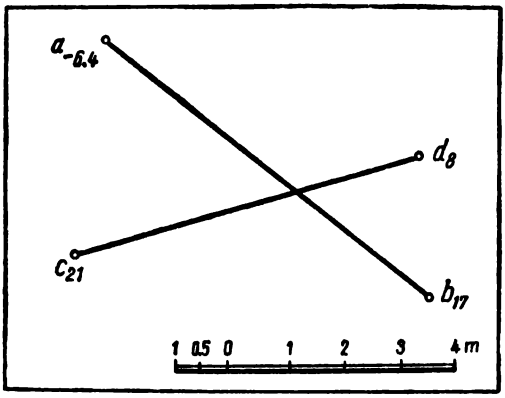

Fig. 409

(see Sec. 79) and compare elevations at the intersection points of the projections. If the elevations prove to be the same for both projections, then obviously the straight lines AB and CD have a common point and therefore intersect. Otherwise the lines are skew lines.

Let us consider two parallel lines AB and CD (Fig. 410); their projections on any plane will be parallel. Also parallel will be their projections on a certain plane V, which is perpendicular to plane H_0

and parallel to the given lines. Hence, the angles which the projections $a'b'$ and $c'd'$ make with horizontal straight lines will be equal: $\alpha_1' = \alpha_2'$. And since $\alpha_1' = \alpha_1$ and $\alpha_2' = \alpha_2$, we have $\alpha_1 = \alpha_2$. The equality of angles of inclination of the parallel straight lines to plane H_0

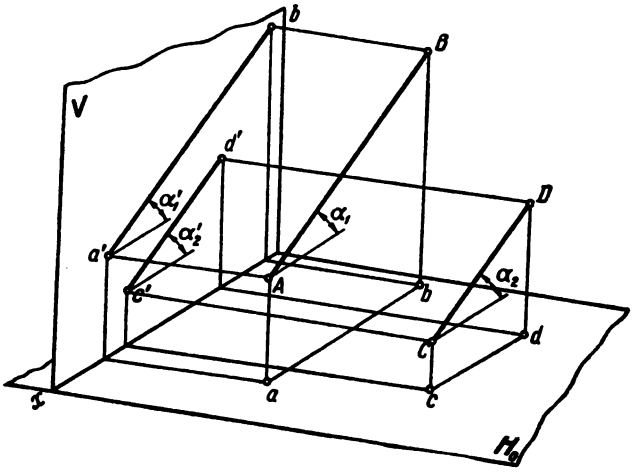

Fig. 410

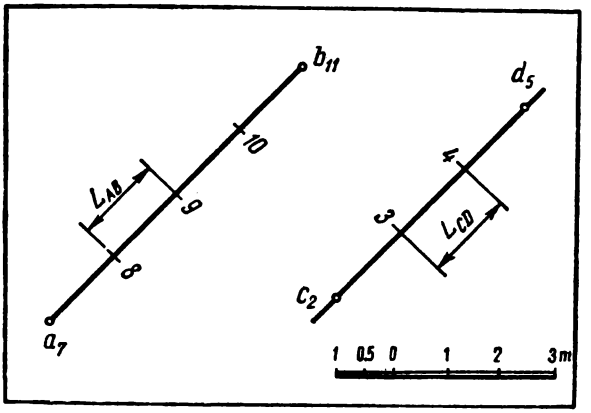

Fig. 411

signifies that they have the same slope and, hence, the same intervals. Thus, in projections with elevations, the projections of parallel straight lines must not only be parallel but also have equal intervals.

It is likewise readily seen (see Fig. 410) that the elevations should increase in one direction.

Fig. 411 gives the projections of two parallel lines AB and CD. Their projections a_7b_{11} and c_2d_5 are parallel, their intervals L_{AB} and L_{CD} are equal and the elevations on the projections increase in one direction.

Sec. 81. The Plane

Aside from the familiar methods of specifying a plane (Sec. 15), the position of a plane, in the case of projections with elevations, is defined by a scale of dip or a scale of slopes of the plane. That is the name given to the graduated projection of the line of maximum inclination of the plane.

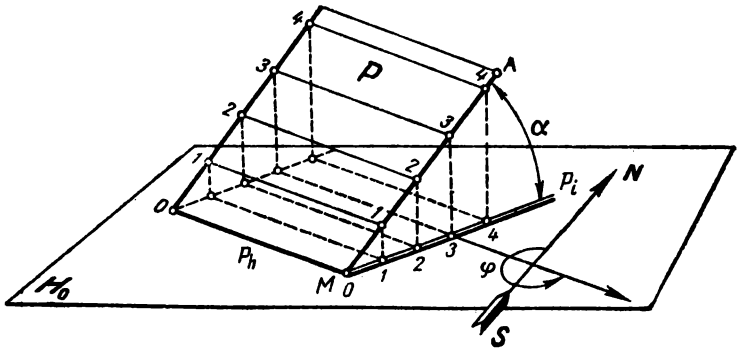

Fig. 412

Fig. 412 gives a spatial image of a plane and its horizontals spaced (in elevation) at 1-metre intervals. The horizontals of the plane and the line of maximum inclination (steepest line) normal to it are projected on plane H_0. The projection of the steepest line is indicated by two parallel straight lines (heavy and light) and is designated by P_i. This is the scale of dip of the plane P. The projections of the horizontals of the plane and the dip scale intersect at right angles (see Sec. 14).

Let us agree to call the slope angle of plane P to plane H_0 the angle of dip α. Angle α is measured by the angle formed by the steepest line AM and its horizontal projection P_i.

In certain cases it is necessary to determine the position of the plane relative to the cardinal points. This is encountered in geological work when determining the mode of occurrence of layers of rock.

For this purpose we introduce the concept of direction of strike of a plane and the angle of strike. By the direction of strike of a

plane we mean the rightward direction of the horizontal lines when looking in the direction of increasing elevations.

The angle formed by a terrestrial meridian and the direction of strike is called the angle of strike φ; this angle is reckoned from the northern end of the meridian counterclockwise to the direction of strike (see Fig. 412).

Example. *Determining the angles of dip α and of strike φ of a plane.* In Fig. 413, plane P is given by three points A, B and C. We graduate segments AB and BC and join with straight lines the points

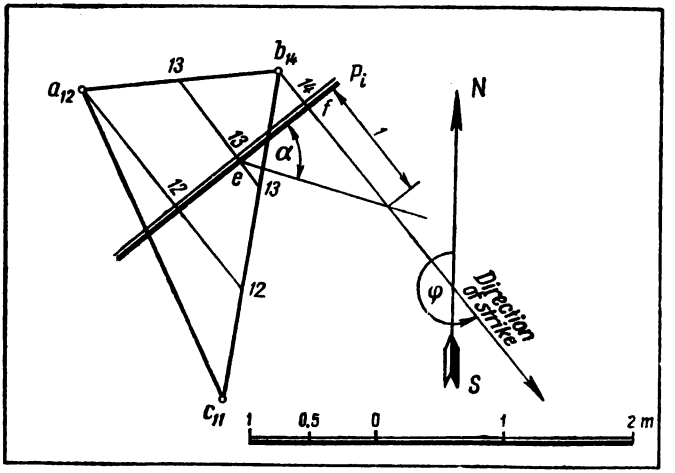

Fig. 413

with identical elevations. These are horizontals of the given plane P; perpendicular to them we draw the dip scale P_i of the plane. To determine angle α, on the dip scale we take two points e and f, the difference of their elevations being unity. Using a right triangle, one side of which is ef and the other a segment equal to unit length, we determine the angle of inclination of the steepest line of plane P to H_0. This angle will be the desired dip angle α.

We then establish the direction of strike; it is indicated by the arrow on horizontal line 14. According to the rule given above, we now construct the angle of strike φ.

Sec. 82. The Relative Positions of Two Planes

1. *Intersecting planes.* To construct the line of intersection of two planes P and R, determine the intersection points of two pairs of their horizontals, each pair with any like elevations. In Figs. 414 and

415, horizontals number 4 intersect in point M and horizontals number 6 in point N. The required line is MN. In Fig. 415 both planes are given by dip scales, perpendicular to which are drawn auxiliary horizontals (numbered 4 and 6). It can easily be shown that if the dip angles of two planes are the same, then the projection of their line of intersection is the bisector of the angle formed by the projections of the horizontals of the planes.

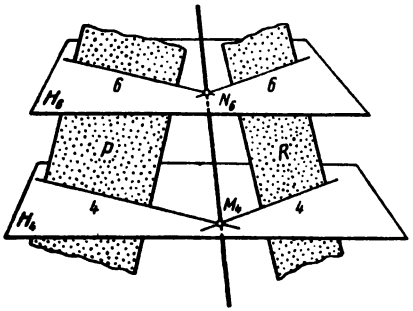

Fig. 414

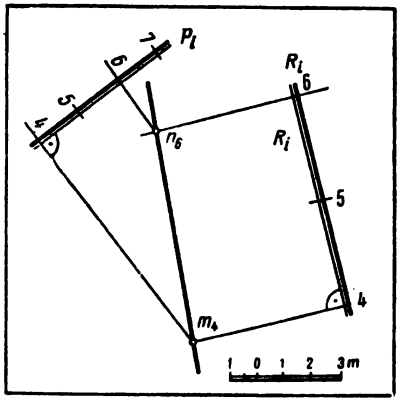

Fig. 415

Fig. 416 illustrates how to find the line of intersection m_3n_2 of two planes P and Q having the same angles of dip. Since the right-angled triangles $a_2m_3n_2$ and $b_2m_3n_2$ are equal (their sides a_2m_3 and b_2m_3 are equal by hypothesis, and the hypotenuse n_2m_3 is common), it follows that the angles at the vertex n_2 are also equal.

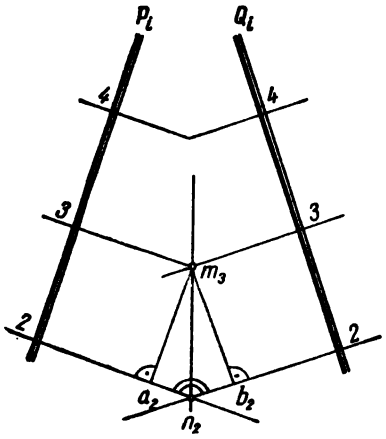

Fig. 416

This statement should be kept in mind when constructing plans for roofs of buildings when the slopes of the roof form identical angles α with the horizontal plane (for examples see Sec. 88).

Let us consider another special case of the intersection of planes P and R when the scales of dip are parallel and the intervals are not equal (Fig. 417). In this case the required line will be a common horizontal (see footnote in Sec. 20) for the construction of which we have to find one point that is common to the given planes. Since the like

horizontals of both planes are parallel, we cannot define the common points of the planes P and R as was done in Fig. 415. To solve the problem we introduce a third, auxiliary, plane S and find the lines of

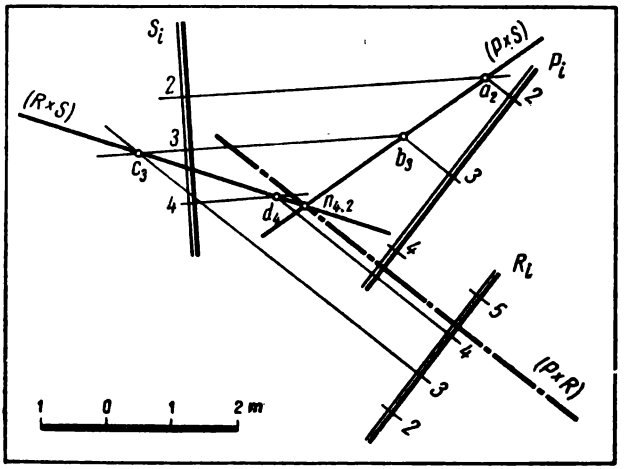

Fig. 417

intersection of planes P and S (straight line a_2b_3), and then of planes R and S (line c_3d_4). The intersection point $n_{4.2}$ of a_2b_3 and c_3d_4 will be a point that is common to the two given planes P and R. Through this point will pass the required intersection line of planes P and R (shown in Fig. 417 as a dot-dash line). All points of this line have the elevation 4.2.

2. *Parallel planes.* A necessary and sufficient condition for the parallelism of two planes is the parallelism of their steepest lines.

Indeed, here the horizontals of the given planes will be parallel (as straight lines perpendicular to the projections of parallel inclined lines). We thus satisfy the general condition for parallelism of planes: two intersecting straight lines of one plane are parallel to two intersecting straight lines of the other plane.

On a projection drawing with elevations the dip scales of parallel planes must be parallel and have equal intervals, and the elevations must increase in the same direction (Fig. 418).

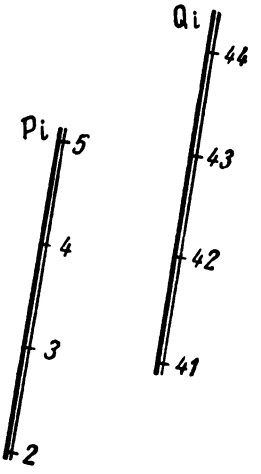

Fig. 418

Sec. 83. The Relative Positions of a Straight Line and a Plane

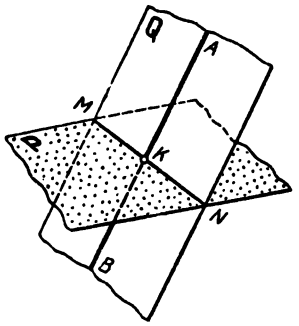

Fig. 419

A line belongs to a plane if two points (for line and plane) have the same elevations. Otherwise the line either intersects the plane or is parallel to it. As in orthogonal projections, the problem of the relative positions of a line and a plane reduces to solving the problem of the relative positions of two straight lines: a given line AB and an auxiliary line MN (Fig. 419), the latter being the line of intersection of the given plane P and the auxiliary plane Q that is passed through AB. Whereas in orthogonal projections a projecting plane was suggested as the auxiliary plane, in projections with elevations an oblique plane should be passed through the straight line. (A horizontal projecting plane is useless because in the drawing the projections of lines AB and MN will coincide.)

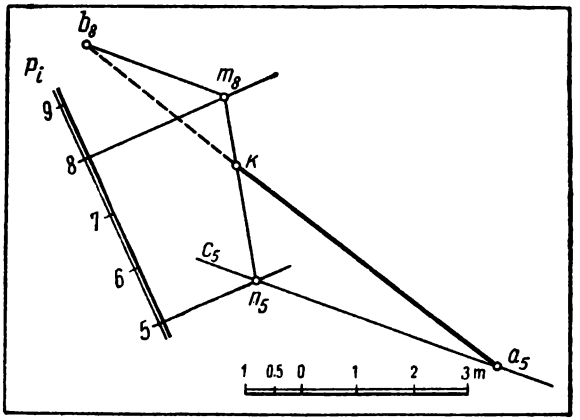

Fig. 420

In Fig. 420 the auxiliary plane is defined by the given line AB and a horizontal AC (a_5c_5), the direction of which is chosen so as to obtain within the limits of the drawing the point N (n_5) which is necessary for the construction. The second point M (m_8) is determined by the intersection of horizontals numbered 8.

Point K, in which ab intersects mn, is the projection of the point of intersection of line AB and plane P. The elevation of this point may be defined by drawing a horizontal of plane P through it.

To summarize then: in order to find the point of intersection of a line with a plane, it is necessary: (1) to draw through the line any oblique plane, (2) to construct the line of intersection of the given plane and an auxiliary plane (line MN), and (3) to determine the required point K as the point of intersection of two straight lines: the given line AB and the constructed line MN.

Sec. 84. A Straight Line Perpendicular to a Plane

We prove a theorem relating to a perpendicular to a plane.

Theorem. If a straight line is perpendicular to a plane, then: (1) its projection is parallel to the dip scale of the plane, (2) the interval of the line is, in magnitude, the reciprocal of the interval of the plane.

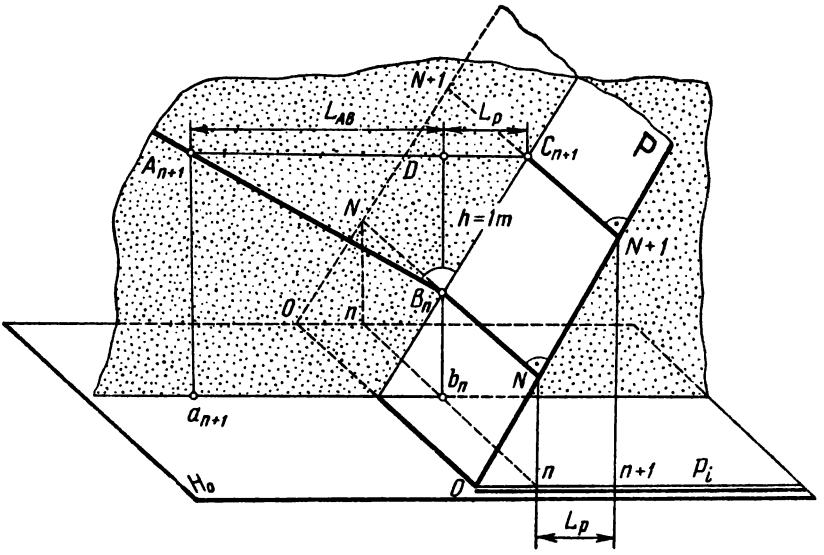

Fig. 421

We refer to Fig. 421, which shows plane P with horizontal lines in it, the dip scale P_i, and a straight line AB perpendicular to plane P.

1. As we know from the section on orthogonal projections, a horizontal projection of a perpendicular to a plane must make a right angle with like (horizontal) projections of the horizontal lines of this

plane; that is, $a_{n+1}+b_n$ is perpendicular to nn, and since we also have P_i perpendicular to nn, it follows that $a_{n+1}+b_n$ is parallel to P_i.

2. We consider the right triangle $A_{n+1}+B_nC_{n+1}$ the altitude of which is dropped from the vertex of the right angle onto the hypotenuse and is equal to one metre. The two sides of the triangle are segments of a perpendicular and the steepest line of the plane, the difference in elevations of the extremities of each side being equal to unity. In this right triangle

$$\frac{L_{AB}}{h}=\frac{h}{L_p}$$

Since $h = 1$, it follows that

$$L_{AB}=\frac{1}{L_p}$$

From Fig. 421 it is evident that L_{AB} is the interval of the perpendicular to plane P, and L_p is the interval of the dip scale of the plane.

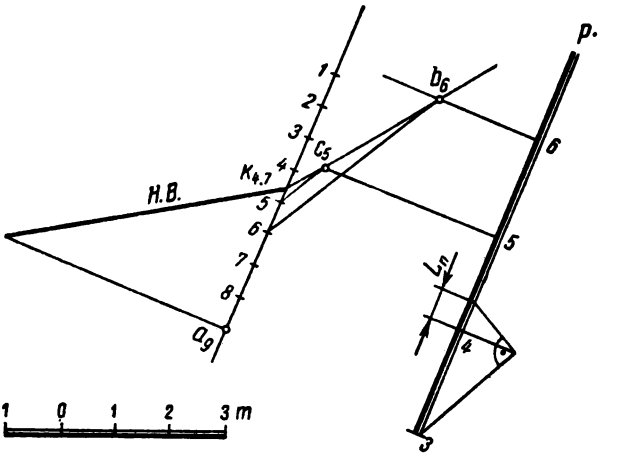

Fig. 422

Note that the projection $a_{n+1}+b_n$ and the dip scale P_i must be graduated so that their elevations increase in the opposite direction.

How this theorem on a perpendicular to a plane may be utilized is shown in the following example.

Determining the distance from point A to plane P given by the dip scale (Fig. 422).

From point A drop a perpendicular on plane P, find point K (the point of intersection of this perpendicular with plane P) and then find the true length of the line-segment AK.

To construct a perpendicular from point A to plane P, draw through a_9 a straight line parallel to the dip scale P_i (see Sec. 80). Having found the interval of the perpendicular L_n, graduate the projection of the perpendicular, remembering that its elevations must increase in a direction counter to that of the elevations of the dip scale.

Through the perpendicular, pass an auxiliary plane (see horizontals 6 and 5) and find the straight line c_5b_6 of intersection of plane P and the auxiliary plane, and then find the point $K_{4.7}$ of intersection of the perpendicular with the intersection line just found. Line-segment $K_{4.7}a_9$ is the projection of the segment that determines the distance from point A to plane P. The magnitude of this distance is found as indicated in Sec. 79.

Sec. 85. The Projections of Solids and Surfaces

When applying the method of projections with elevations for depicting geometric solids, it is necessary to indicate on the horizontal projection of the solid the elevations of characteristic points and lines (if the entire line has one elevation). The characteristic points of a polyhedron are its vertices.

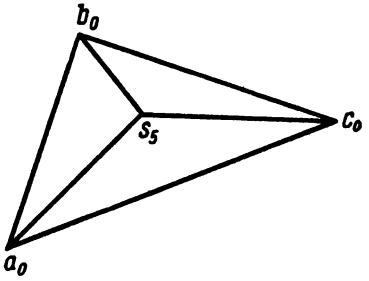

Fig. 423

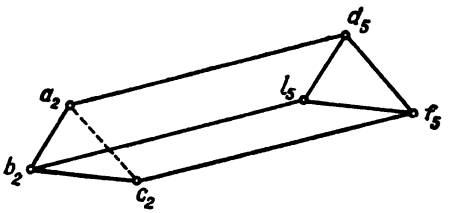

Fig. 424

In Fig. 423 we have a pyramid depicted in projections with elevations; the base lies in the H_0 plane and the vertex is five metres from the plane. Fig. 424 illustrates a trihedral prism, the lower base of which lies in the plane H_2 and the upper base of which is cut by a plane with the elevation 5.

If a solid is bounded by a curved surface, it will be necessary, when depicting it in projections with elevations, to resort to horizontals, which in these cases will be the lines of intersection of the surface of the given solid with planes parallel to the H_0 plane.

In Figs. 425 and 426 we see an oblique cone and an oblique cylinder. Their horizontals are circles with centres lying on the axes of the solids. In some cases, special constructions are required to draw the horizontals of the surface. An instance is the drawing of the hori-

zontals of a surface of identical slope; this surface is the envelope of a family of right circular cones, the vertices of which lie on a certain space curve AB (Fig. 427). The axis of each cone of the family is vertical. The envelope of such a family of cones is a ruled surface,

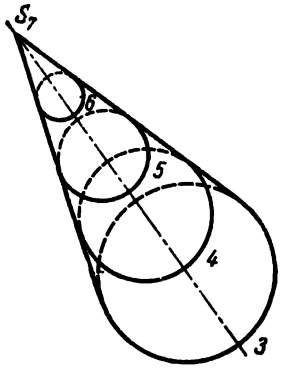

Fig. 425

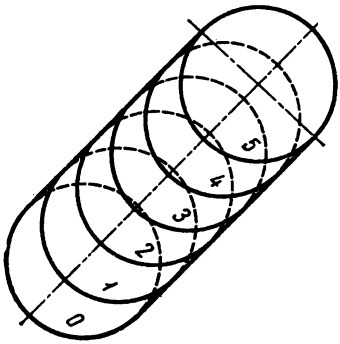

Fig. 426

all the straight-line generatrices of which make with the H plane identical angles, equal to the slope angle of the generatrices of the cone to H.

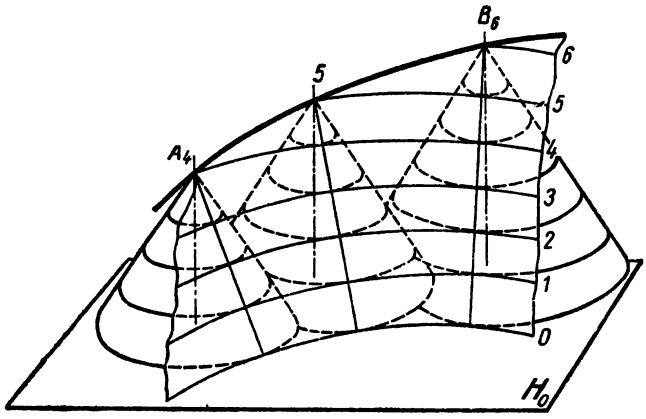

Fig. 427

Fig. 428 illustrates the construction of the horizontals of a surface of identical slope on a drawing in projections with elevations. Here, each horizontal of the surface is an envelope of a family of the

horizontals of cones, all the horizontals of a given family having the same elevation. For example, in Fig. 428 the horizontal of the surface with elevation 3 envelopes the family of horizontals of cones having the same elevation 3.

A special case of a surface of identical slope is an evolute screw surface (see Sec. 42). One of its applications (building the embankment lines of fills and cuts of railway beds on curves and inclines) will be discussed in detail later on.

In conclusion let us examine the depiction of the surface terrain of the earth by means of horizontals (contour lines). The surface terrain of an area may serve as an instance of what is called a topographic surface, the formation of which does not obey any geometric law.

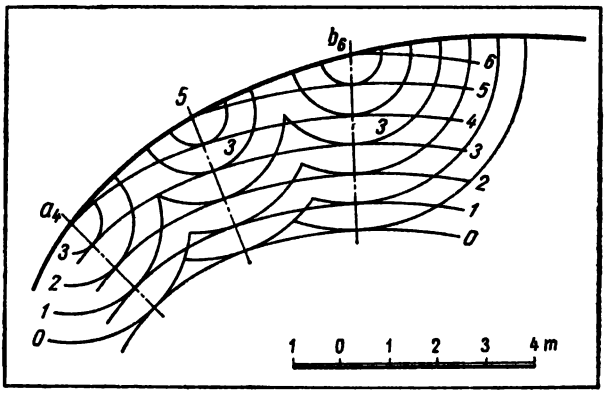

Fig. 428

The distance between horizontal planes that intersect the topographic surface is taken equal to one or several metres. The elevations of the horizontal lines (contour lines) help one to visualize the terrain depicted on the plan (map). In the centre of the map shown in Fig. 429 is a hill (the highest contour line has the elevation 100); in the right upper corner is a valley with the lowest contour line at 92. In addition to the elevations, representation of terrain involves so-called bergstrichs, which indicate the direction of surface slope. In the case of hills, the bergstrichs are marked on one of the upper contour lines in the direction of water run-off. In depressions the ends of the bergstrichs are directed inwards from the lowest contour line of the terrain.

When solving problems on a topographic surface, it is assumed that a straight line (say AB or CD in Fig. 429) joining two points of adjacent contours is located on the surface at all points. It is obvious

that the smaller the difference between contours, the smaller the error in this assumption. Of all straight lines connecting a given point A of a surface with points of an adjacent contour line, the one with steepest slope will be that whose horizontal equivalent is a minimum.

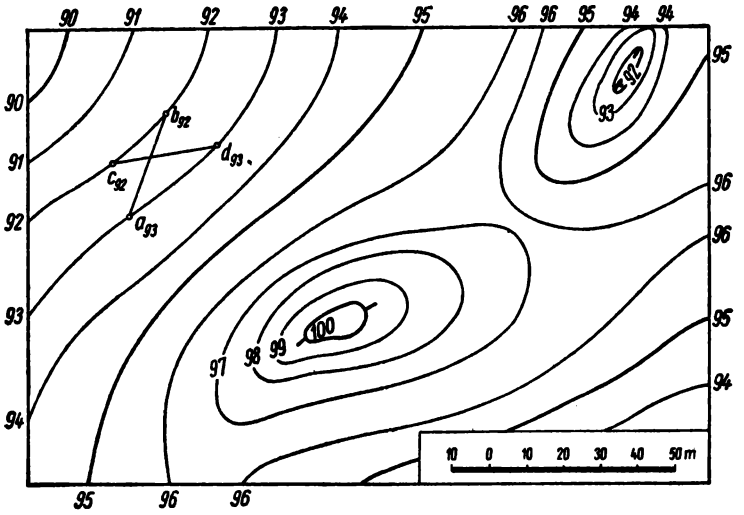

Fig. 429

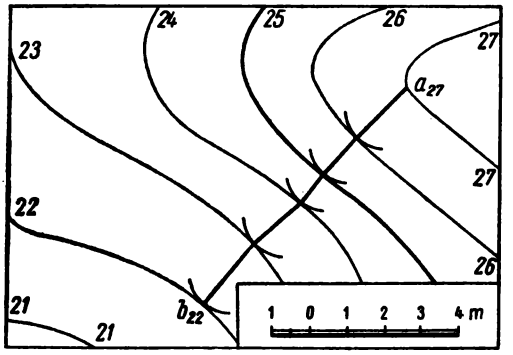

Fig. 430

In Fig. 430 we have constructed a steepest line drawn through point A on a topographic surface. To do this, from A as a centre strike an arc that is tangent to the closest (26th) contour; from the point of tangency strike a second arc tangent to the next (25th) contour, and so forth. We obtain the desired line by joining the points of tangency.

Sec. 86. Intersection of Surfaces

Since each surface (including a plane surface) is depicted by means of a family of horizontal lines (contour lines), the line of intersection of the surfaces may be constructed as a locus of intersection points of horizontals (contours) with identical elevations. An elementary instance is the construction of a line of intersection of two planes considered in Sec. 82.

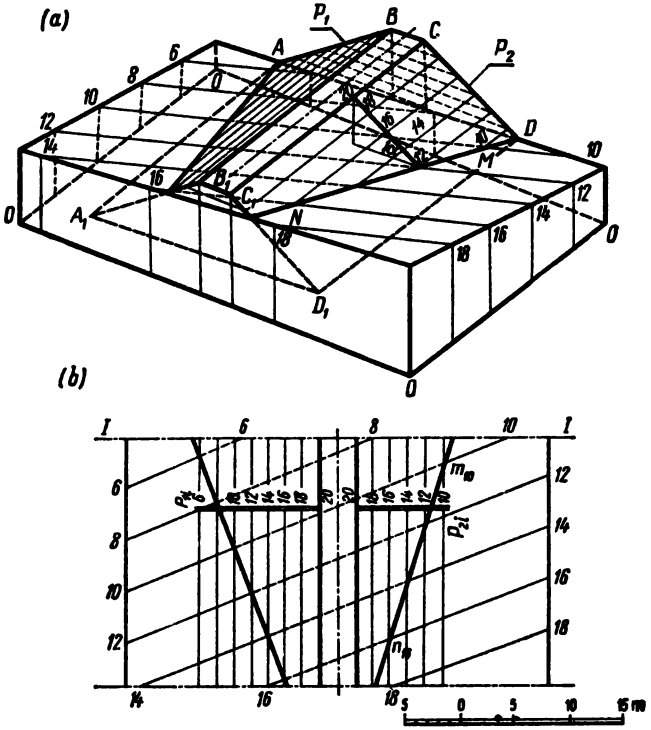

Fig. 431

More involved cases of the intersection of surfaces are considered below.

Example 1. *The intersection of embankments of a railway bed fill with a plane declivity.*

(1) Let the axis of the bed be rectilinear and horizontal (Fig. 431). The earthen bed in this case is a prism, the bases of which are normal cross-sections (profiles) of the fill $ABCD$ and $A_1B_1C_1D_1$ (Fig. 431*a*).

The inclined faces P_1 and P_2 of the prism form with plane H_0 an angle α, the tangent of which (according to specifications) is 1 : 1.5.

Hence, the interval of the dip scale of planes P_1 and P_2 (the faces of the embankments) will be 1.5 m. Noting that in this case the edges of the bed are horizontal, we construct the scales of dip P_{1i} and P_{2i} and then the contour lines (Fig. 431*b*) of the embankments of the fill and determine the points of intersection of the contours of the embankments with like contours of the plane declivity (it is sufficient here to determine two points, say M and N, since the line of intersection will be straight). The straight lines thus found will be the boundaries of the fill; dumping will begin at these points.

(2) Let us consider the case when the roadbed is rectilinear but not horizontal. Assume that the bed has an incline $I = \frac{1}{22}$, the slope of the embankments of the fill and cut is $i = 1 : 1.5$; the road axis at point A (Fig. 432) has the elevation 47 m. To determine the boundary line of earthwork in constructing the railway bed we have to construct the contour lines of the embankments of fill and cut. This requires solving the following problem (Fig. 433*a*): through a given straight line AB to pass a plane P, the inclination of which is i. The solution reduces to constructing a plane tangent to a cone whose vertex lies in one of the points of the given line, the generatrices of the cone having a given inclination of i $\left(i=\frac{h}{R}\right)$.

For height of cone $h = 1$ m, the radius of the base, $R=\frac{1}{i}$.

A line tangent to the base of the cone and drawn from point B, in which the given straight line intersects the plane of the base, is the horizontal line (contour) of the desired plane (Fig. 433*a* shows one of two planes that satisfy the requirements of the problem). The line of tangency (the generatrix AK) is the steepest line of the constructed plane, and its graduated projection will be the scale of dip. In Fig. 433*b*, we have the solution of this problem in projections with elevations.

We return to Fig. 432 where the foregoing solution is executed four times. Here the edges of the fill embankments and the lines CD and EF serve as straight lines through which we pass the planes of the given inclination (two lateral planes of the fill and two of the cut). Lines CD and EF lie in the plane of the bed but are displaced two metres from the edges for the embankments of the cuts. The extra two metres on either side of the bed axis in the zone of cut is for drain ditches. The elevations of each shoulder are obtained by means of contour lines of the roadbed; the latter are drawn perpendicular to the axis of the bed through points found by graduation of the axis.

A comparison of the elevations of the terrain and the roadbed shows that a fill is required on the southern side of the section and a cut in the northern part.

To construct the contour lines of the eastern embankment of the fill, we take two points (elevations 44 and 46) on the road edge. (Note

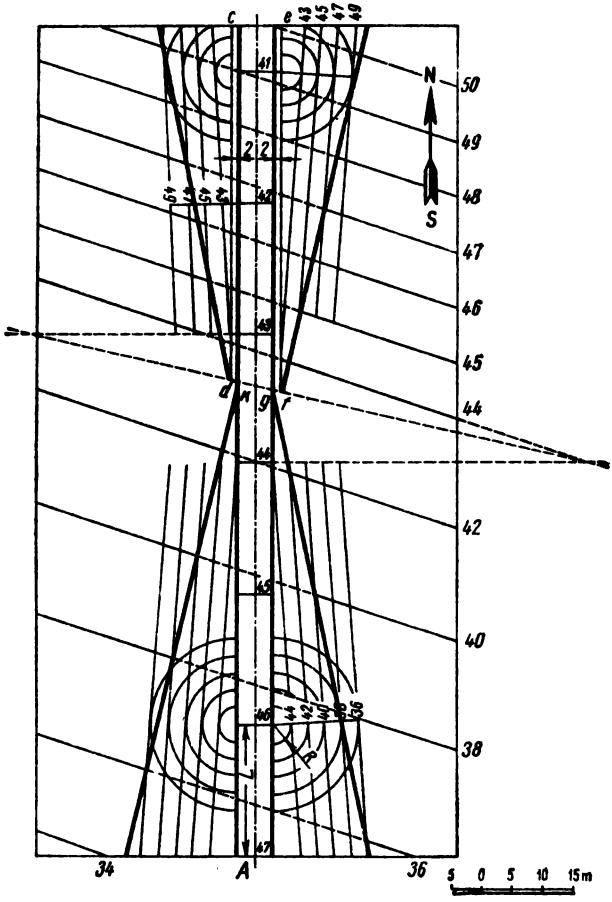

Fig. 432

that in this case the edges are not horizontal lines, as in the case in Fig. 431.) We put the vertex of the auxiliary cone in the second of the two points. For a two-metre (46 — 44) altitude of the cone, the radius R of its base must be 3 m $\left(R=\frac{h}{i}\right)$. The cone base will be

located in one horizontal plane with point 44 taken on the road edge.

A line tangent to the circle of the base of the auxiliary cone drawn from point 44 is the 44th contour line of the eastern embankment of the fill. Parallel to this line, at a 3-metre separation, we draw even contour lines of the fill embankment. They will obviously be tangent to appropriate contour lines of the conic surface (if it is extended downwards). The scale of dip of the embankment plane will be perpendicular to the contour lines of the embankment, but not to the

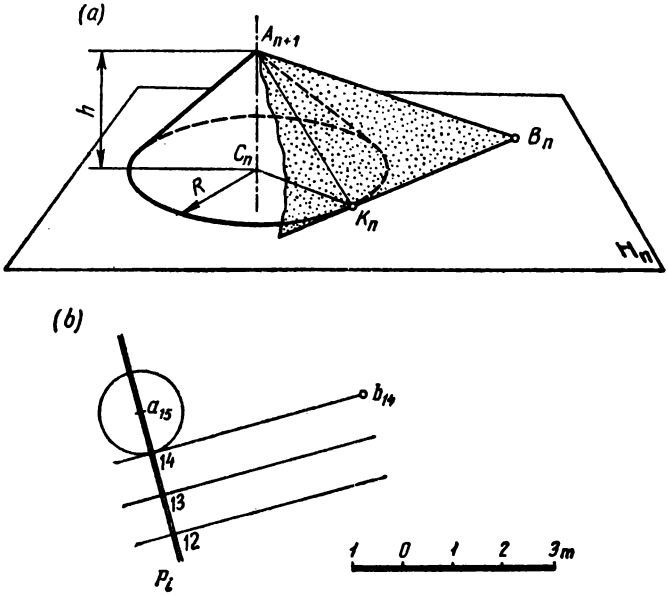

Fig. 433

edge of the roadbed. We find the boundary of the fill on the east as a straight line of intersection of two planes: the lateral surface of the fill and the surface of the declivity.

In the very same way we find the boundaries of the embankments of cut, only we put the vertex of the auxiliary cone downwards, locating it in the point with an elevation lower than that from which the tangent to the cone base is drawn.

Points *K* and *G* (which lie on the edges of the roadbed), like points *D* and *F* (which lie on the contour of the bed extended in both directions from the road axis), must lie on a single straight line. The plane of the road intersects the plane declivity along the straight

line DF, which defines the line of transition from fill to cut; it is called the line of zero earthwork.

Example 2. *The section of an inclined cone by a plane P given by the scale of dip.*

The cone base has the elevation 2, the vertex, 10 (Fig. 434).

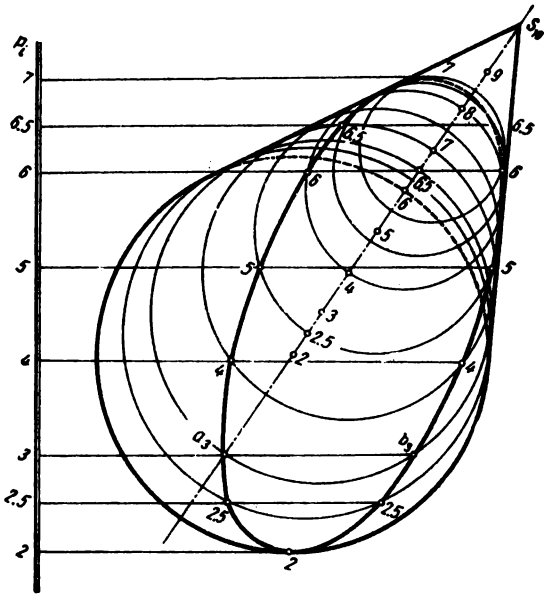

Fig. 434

The required line is the locus of intersection points of horizontal lines of the cone (circles) and a plane (straight lines). For instance, points A and B are obtained by intersection of horizontal lines (contours) with the same elevation 3.

Sec. 87. The Intersection of a Straight Line with a Surface

The intersection of a straight line with a plane is considered in Sec. 83.

When a straight line intersects a geometric solid, it is required to find several (mostly two) points of intersection.

1. *The intersection of a straight line MN with a prism.* Through a given straight line (Fig. 435) we pass an auxiliary elementary cutting plane (see Sec. 36) defined by the straight lines MN and NS; draw NS

parallel to the lateral edges of the prism (the conditions for parallelism of two straight lines are given in Sec. 80). The elementary cutting plane will intersect the base of the prism (lying in plane H_3) along the line MS, and the lateral faces of the prism along two straight lines PK and QL, which are parallel to the lateral edges of

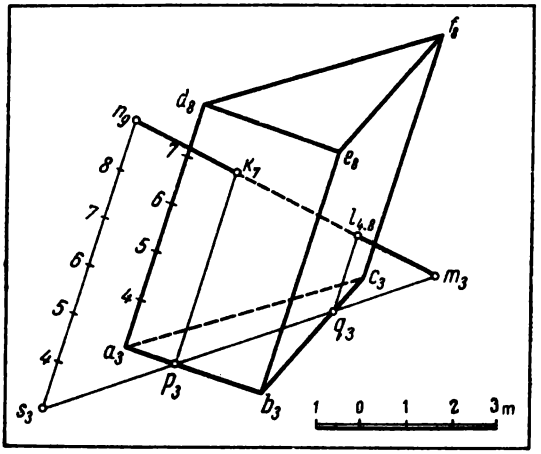

Fig. 435

the prism. The latter two lines intersect the given line MN in points K (k_7) and L ($l_{4.8}$), which will be the desired points.

Similarly we can find the points of intersection of a straight line with the surface of a cylinder.

2. *The intersection of a straight line MN with a pyramid.* Through a given straight line (Fig. 436) we pass an auxiliary elementary cutting plane defined by two intersecting straight lines MN and MS (line MS is drawn through the vertex of the pyramid). This auxiliary plane will intersect the pyramid base (lying in plane H_2) along the straight line PN, which is parallel to the horizontal line MS, and the lateral faces of the pyramid along two straight lines PS and QS. Points K and L, in which PS and QS intersect the given line MN, will be the desired points, common to the given straight line and the surface of the pyramid; in other

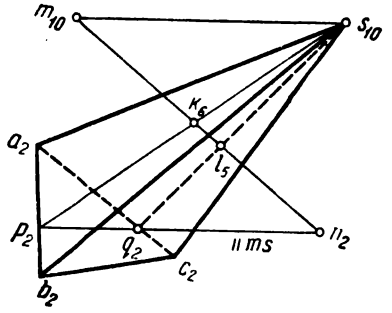

Fig. 436

words, the points of intersection of the line with the pyramid.

The intersection of a straight line with a cone may be found in the same way.

3. *The intersection of a straight line with a topographic surface.* In this case too (Fig. 437), we graduate a given straight line AB and

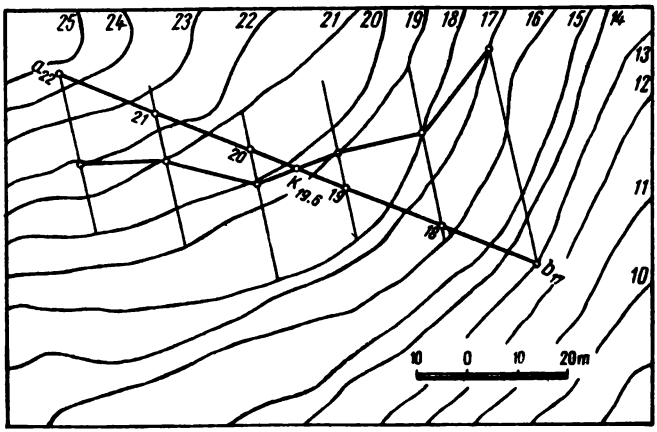

Fig. 437

pass an auxiliary plane through it. Then we determine the intersection points of like horizontals (contours) of the plane and the topographic surface. The locus of the points thus found is the line of intersection of the plane and the surface. And point K, in which the given straight line AB cuts the intersection line thus found is the point that is common to the given straight line and the topographic surface.

Sec. 88. Examples from Engineering Construction

Example 1. *Determining the boundary lines of a fill and a cut at a construction site.*

Let it be required to determine the boundary lines of earthwork for building a horizontal construction site at an elevation of 51 metres, the contour of which is shown in Fig. 438.

The slopes of the embankments are: of the cut, $i_1 = 1 : 1$, of the fill, $i_2 = 1 : 1.5$. The straight-line access road to the site has a slope of $I = 1 : 4$.

The line of zero-work operations lies on the 51st contour of the terrain. The northern part of the site will lie in a cut, the southern part on a fill. In the cut area we expand the site round the entire

perimeter for the construction of drain ditches. The sought-for boundary lines of earthwork construction will be the lines of intersection of the topographic surface with the embankments of the fills and cuts. To find these lines we have to construct the contours of all sloping

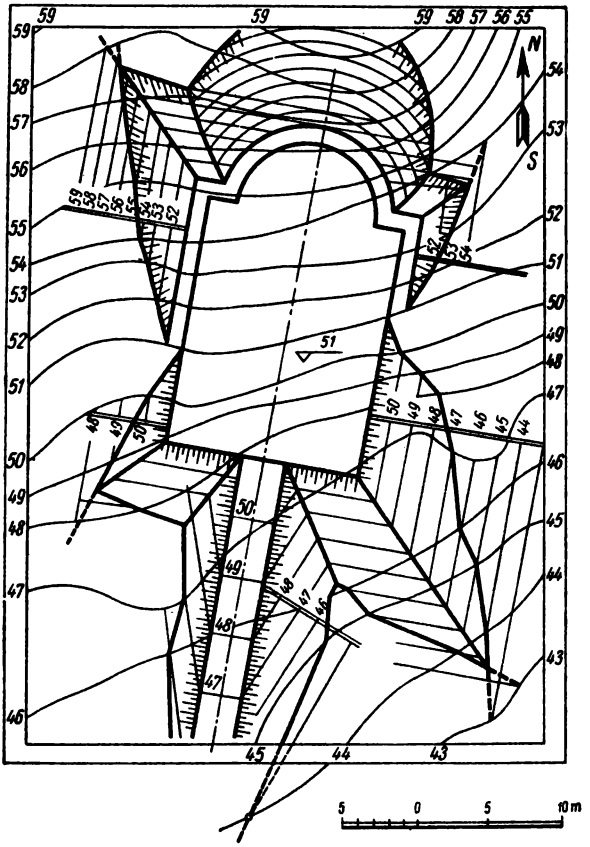

Fig. 438

sides (embankments). The distance between contours of the cuts will be one metre $\left(L_1=\frac{1}{i_1}\right)$, and for fills: 1.5 m $\left(L_2=\frac{1}{i_2}\right)$. Temporarily disregarding the semicircular projection and access road, we construct the contours of three fill embankments and three cut embankments. The contours of the plane embankments will be straight lines parallel to the corresponding sides of the construction site.

The surface of the cut adjoining the northern semicircle is a conic surface, the horizontals (contour lines) of which will be concentric arcs of circles.

The contour lines of the plane embankments of the access road are constructed with the view that they pass through inclined road edges (see Sec. 86, Fig. 433).

As a result, each surface of the embankments of cut and fill is now depicted by its own contour lines. It now remains to construct the lines of intersection of these surfaces with the earth surface and with each other. The required lines are determined by the points of intersection of like horizontals (contours) of the surfaces under consideration.

Example 2. *Determining the boundary lines of fill and cut on a section of railway on an upgrade curve.*

The contour map in Fig. 439 specifies the axis of the railway bed and the elevation of a point A (a_{26}) on the axis. The roadbed from A to B has an upgrade of $i^0/_{00}$. Also given are the normal cross-sectional profiles of the fill and cut.

It is required to determine the boundary lines of earthwork in constructing the roadbed.

The following is a recommended solution of this problem.

1. Determine the interval L $\left(L=\frac{1}{i}\right)$, graduate the axis of the railway and draw contour lines of the railway bed. To graduate the curve, take 1/nth part of the interval L with compass and strike n arcs on the axis from point A. This yields point 27 on the axis. Continuing this operation, we get other elevations. Ordinarily, $n=2$ to 4. The contour lines of the railway bed will be straight lines, the horizontal projections of which intersect in a single point: the centre of the axis of the railway. As was demonstrated in Sec. 43, the surface of the railway bed on an upgrade curve will be a right helicoid (a particular case of a conoid).

2. Comparing the elevations of the axis and topographic surface, we establish the sites of cut and fill. Around a cut, the roadbed surface is widened two metres on both sides of the axis for the construction of drain ditches. To do this, draw lines CD and EF, the projections of which are concentric to the edges of the railway bed*.

3. Construct the contour lines of a family of right circular cones whose vertices are located on the edges of the railway bed in the case of a fill, and on the lines CD and EF in the case of a cut. The vertex of one of the auxiliary cones lies in point 28 on the outer edge. The label of the cone vertex is 28 m. The radius of the first circle of a

* These are arbitrary since there are no corresponding lines on the embankment of the cut.

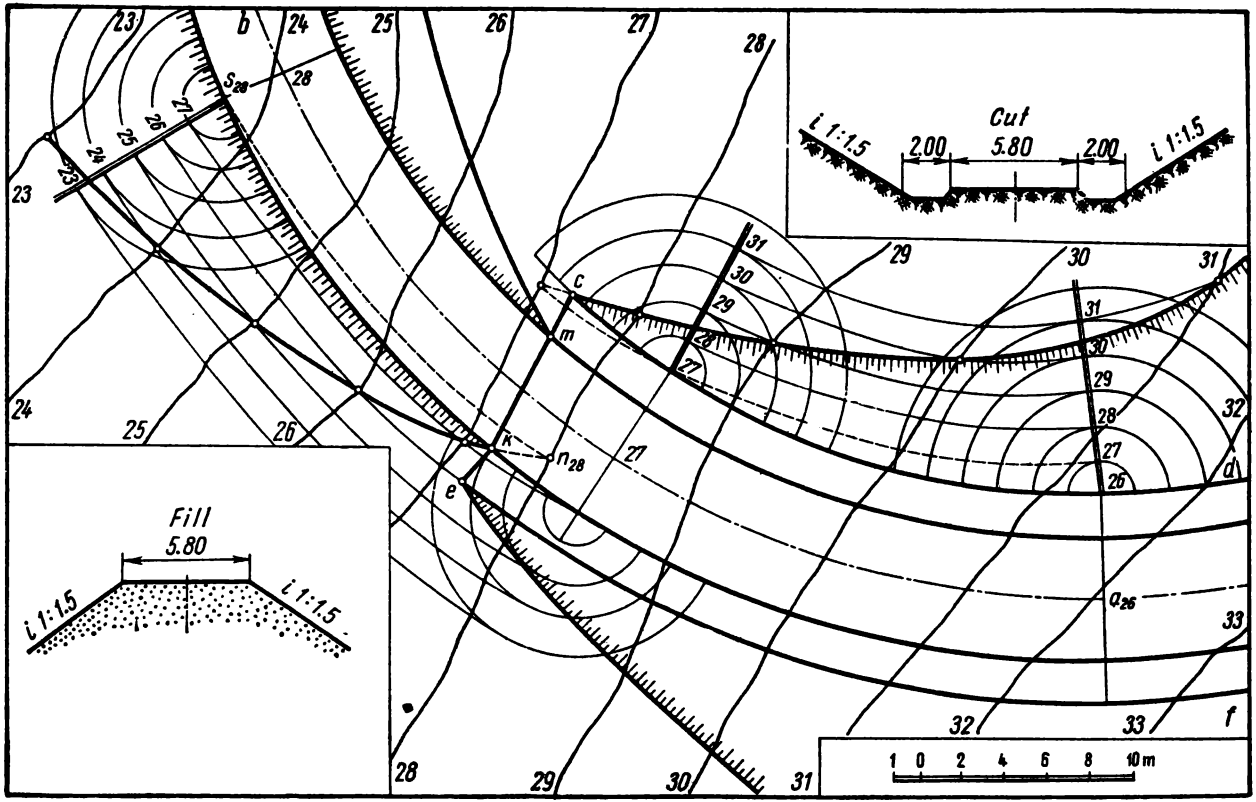

Fig. 439

cross-section of the cone is equal to the interval of the embankment, which is 1.5 m, the elevation of the circle is 27 m. The radii of subsequent circles increase each time by the magnitude of the interval, and the elevation diminishes by unity. Similarly we construct the horizontal lines (contour lines) of a cone whose vertex lies on the outer edge at a point with the elevation 27. The horizontal lines of the family of cones for the embankment of a fill adjoining the inner edge are not shown in Fig. 439.

In the cut, the vertices of the auxiliary cones should point downwards. The elevations of the horizontal lines of each cone will increase as the radius of the circle increases. In Fig. 439 we have the horizontal lines of only two auxiliary cones whose vertices lie on line CD at points with the elevations 26 and 27 m.

4. Draw the horizontal (contour) lines of the embankments of fill and cut. Each contour line of the embankment will be the envelope of a family of circles with the same elevations.

5. Determine the boundary lines of the earthwork. Each of the constructed boundaries is the locus of points of intersection of contours of embankment and earth surface with identical elevations.

Of particular interest is the construction of points defining the line of zero-work operations. Points K and M on this line (on the edges) are obtained via intersection of the edges with the boundary lines of the fill embankments. For this purpose, to the points of intersection of the contour lines of the earth and the fill surface (which are located outside the railway bed) we have to add another point, N (n_{28}), whose projection lies on the other side of the road edge. In turn, the construction of point N required drawing an additional contour line of the fill embankment with elevation 28.

Analogously, we find points C and E, in which the boundary lines of the cut intersect the contour of the widened railway bed.

Example 3. *Constructing the plans of roofs of buildings*, when the roof slopes form identical angles with the horizontal plane.

In such cases the lines of intersection of the roof slopes are bisectors of angles between the horizontals of the intersecting slopes (see Sec. 82).

Fig. 440 gives the plan of a four-slope roof, where the horizontals of the slopes are the cornices CD, DE, EF, and FC. They are all in a single horizontal plane and have the elevation n. The bisectors of the right angles (straight lines AC, BD, BE and AF) are the lines of intersection of the respective slopes. Line AB is the ridge.

In the same drawing, two slopes are coincident with the plane of the cornices. To bring into coincidence slope ACF, it was necessary to determine the radius of rotation of point A. This radius is the hypotenuse, which may be constructed if we know the angle α or the elevation difference $(h - n)$.

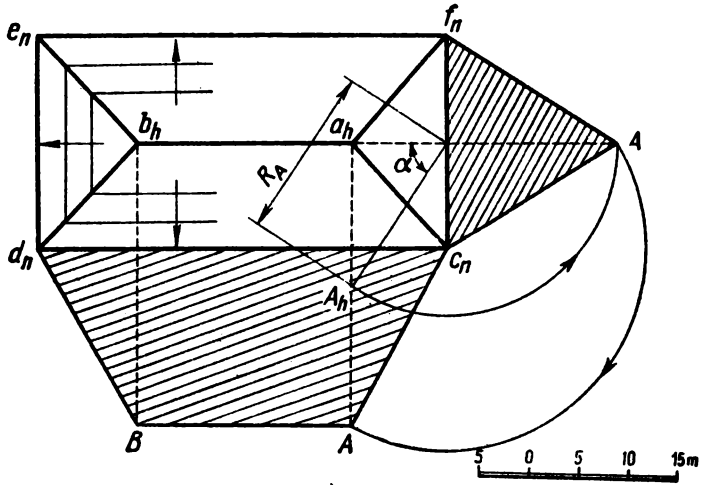

Fig. 440

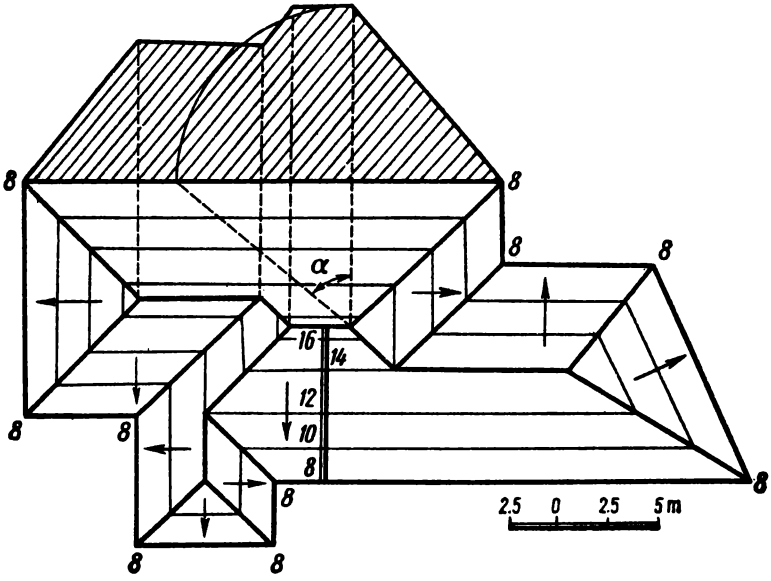

Fig. 441

To bring to coincidence the second slope, we draw lines a_hA and b_hB perpendicular to the axis of rotation: the cornice CD, and on the first one we struck an arc of radius c_nA from point c_n as centre.

An instance of constructing a roof with a more intricate configuration is given in Fig. 441. The principle for determining the lines of intersection of the slopes remains the same. To bring one of the slopes of the roof into coincidence with the horizontal plane, we have to know either angle α or the dip scale of any one of the slopes.

Exercises

1. Given: the projection of a segment of the straight line AB (Fig. 442). Determine: (1) the true length of the segment and angle α; (2) the point with zero elevation.

2. Establish the relative positions of two straight lines given by the projections of their segments AB and CD (Fig. 443).

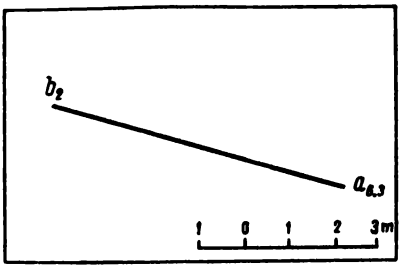

Fig. 442

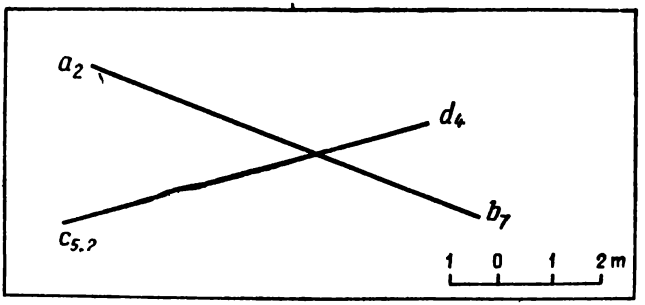

Fig. 443

3. Through a given point A (Fig. 444) pass a plane P, the angle of dip of which $\alpha=30^\circ$ and the angle of strike $\varphi=45^\circ$.

4. In plane R (Fig. 445), which is given by the scale of dip, draw through point A straight lines with the following slopes: $i_1=1:2$, $i_2=1:3$, $i_3=2:1$. For what value of i is the problem solvable?

5. Through point A draw a straight line that will intersect a given straight line BC and will be parallel to plane P given by the scale of dip (Fig. 446).

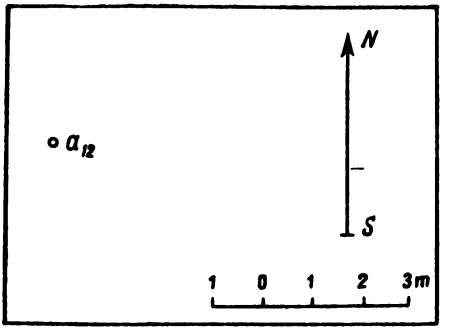

Fig. 444

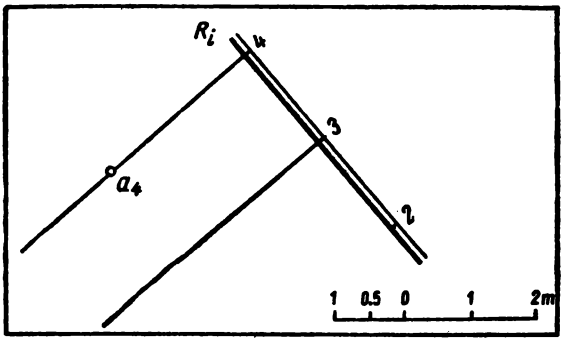

Fig. 445

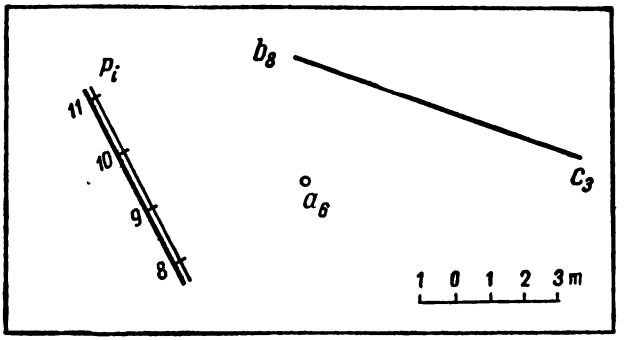

Fig. 446

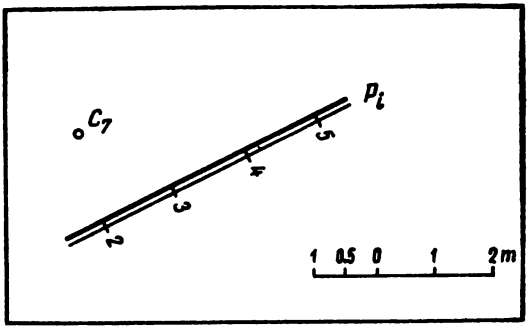

Fig. 447

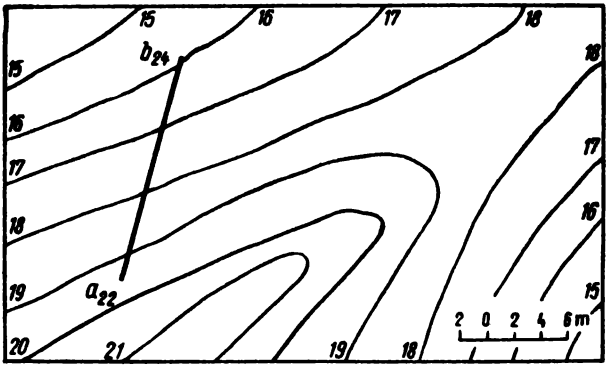

Fig. 448

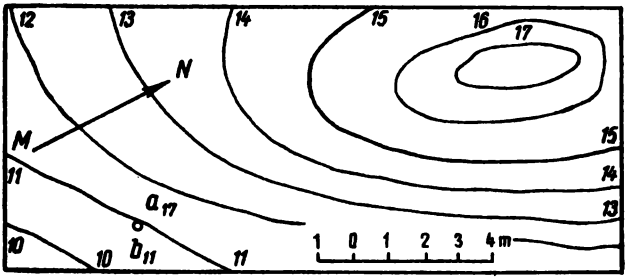

Fig. 449

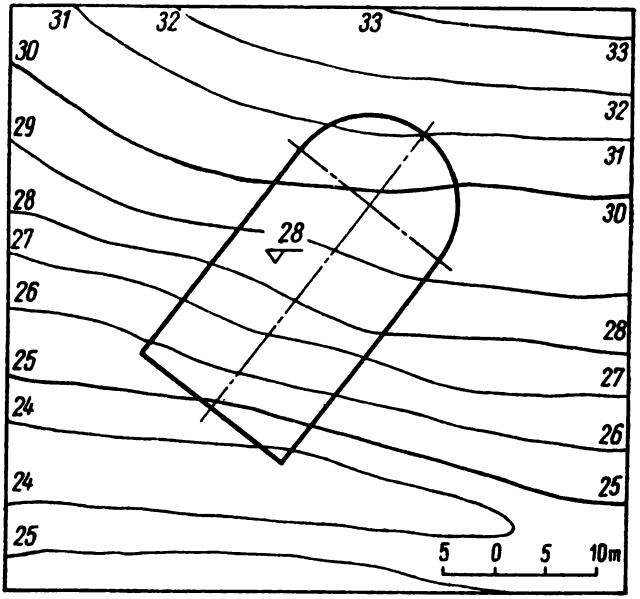

Fig. 450

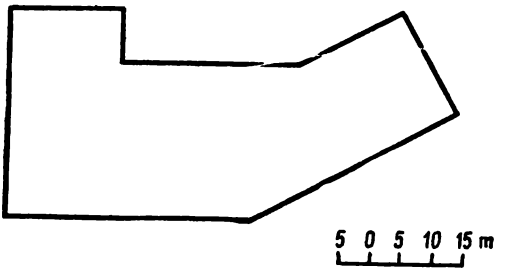

Fig. 451

6. Construct the projection of a sphere if we know the centre of the sphere C and a plane P tangent to the sphere (Fig. 447).

7. Construct the line of intersection of a topographic surface with a plane having a slope $i=1:2$ and passing through a given straight line AB (Fig. 448).

8. Construct the shadow of a vertical pole AB on a topographic surface if the pole is 6 metres high and the light rays make an angle of 45° with the horizon. In Fig. 449 the arrow MN indicates the direction of the ray projections.

9. A horizontal construction site, the contour of which is shown in Fig. 450, has the elevation 28. Construct the lines of intersection of the embankments of fill and cut with the topographic surface. The slope of all embankments is $i=1:2$.

10. Construct the lines of intersection of slopes of a roof, the contour of which is shown in Fig. 451. The number of slopes is equal to the number of sides of the given contour, and all slopes have the same inclination.

Part Three

SHADES AND SHADOWS IN ORTHOGONAL PROJECTIONS, PERSPECTIVE AND AXONOMETRY

*

CHAPTER XVI

THE GEOMETRIC FUNDAMENTALS OF THE THEORY OF SHADES AND SHADOWS

Sec. 89. General Concepts

To make a plane drawing more expressive and impart more pictorialness to a two-dimensional representation, we make use of shades and shadows, which are widely employed in architectural work (facade views of buildings and other structures).

The basic problem in this theory is determining the contours of shades and shadows of a given body.

Shades are produced on the unilluminated surface of the body itself; shadows are cast by the object onto the projection plane and also onto other objects.

When constructing shades and shadows we assume that: (1) the light source is at infinity; (2) light is propagated in straight lines; (3) the light rays are mutually parallel.

In most cases the direction of light rays is that of one of the diagonals of a cube, two faces of which coincide with the planes of projection (Figs. 452 and 453). The projections of a diagonal of a cube will be the diagonals of squares, which means that each of the projections of a light ray makes an angle of 45° with the x-axis.

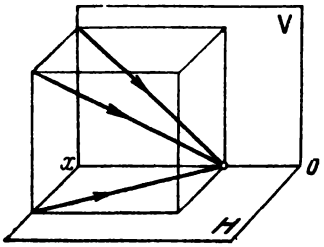

Fig. 452

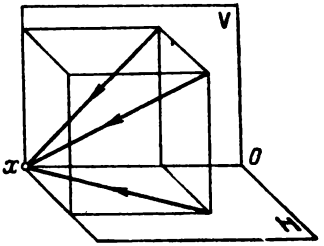

Fig. 453

If the light rays are parallel, the illumination is called sun illumination, in contrast to torch illumination when the source of light is only a short distance from the object and the light rays form a beam of intersecting straight lines.

Sec. 90. The Shadow of a Point

Let there be given: a point A located in the first octant and the direction of light rays (Fig. 454). One of the light rays will encounter point A and will be stopped. The shadow of the given point will be located at the point where the ray, when extended, intersects the projection plane. In other words, the shadow of a point is the trace of a light ray passing through the given point. Its shadow will be located on the projection plane first encountered by the ray. Thus, in Fig. 454 plane H is intersected by a light ray in point A_H before plane V is. For the light ray, A_H is the horizontal trace, and for point A, through which the ray passes, it is its shadow on plane H. Similarly, point A_V is the vertical trace for the light ray, but for point A it is its shadow on the V plane.

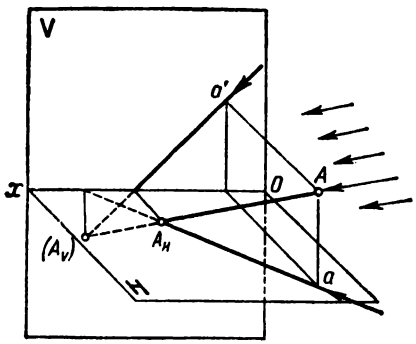

Fig. 454

Of these two shadows, the first (A_H) will be the real one, the second (A_V) the imaginary one. The shadow of the point on the H plane is real because here the ray intersects the H plane before it does the V plane. Let us agree to designate imaginary shadows in parentheses. Figs. 455 and 456 illustrate the construction of the shadows of a point in orthographic representation. In the second example, the real shadow will be

that of the point on the *V* plane.

Obviously, only points located in the first octant can have real shadows.

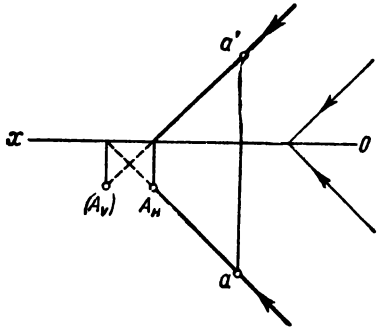

Fig. 455

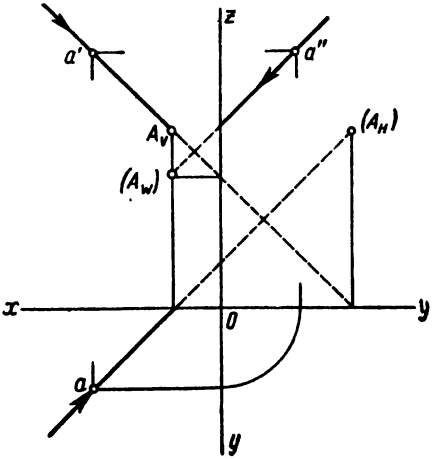

Fig. 456

Sec. 91. The Shadow of a Straight Line

To construct the shadow of a straight line on a projection plane, it is necessary to determine the shadows of any two of its points. The straight line joining these points will be the shadow of such a line

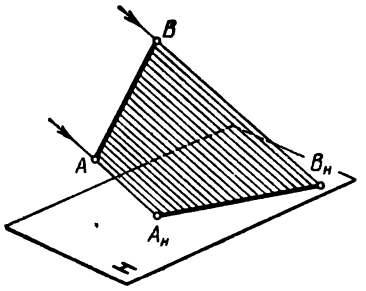

Fig. 457

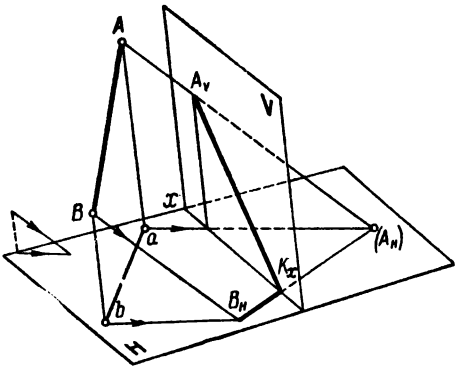

Fig. 458

(Fig. 457). At the same time we can regard A_HB_H as the trace of the radial plane that passes through the given straight line AB.

It is advisable to construct the shadow of a line-segment on two projection planes in the following sequence.

1. Construct the shadow of the segment on one of the projection planes on the assumption that the other does not exist. In the examples given in Figs. 458 and 459, we first construct the shadow of the segment on the H plane.

2. If the constructed shadow intersects the x-axis, then the shadow will break in this point and pass from one projection plane to another.

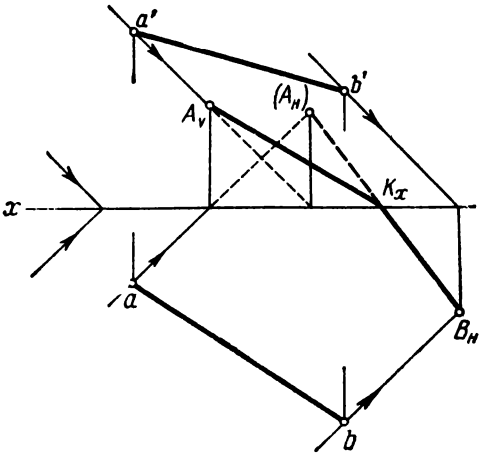

Fig. 459

The point of deflection of the shadow is here designated as K_x. Having established which of the two shadows of the end points of the segment is imaginary, we find its actual shadow on the second projection plane.

It is to this point that the deflected shadow of the line will be directed. In Figs. 458 and 459, the real shadow A_V is just such a point.

3. If a line-segment is located in different octants, we must first choose that part which lies in the first octant. To do this, we have to find the traces of the given segment.

Sec. 92. The Shadow of a Plane Figure

Let there be given a plane opaque triangular plate (Fig. 460). To construct its shadow we have to construct the shadows of all its sides. The shadow of the perimeter of the triangle on one projection plane will also, in the general case, be a triangle. The entire area inside this contour $A_HB_HC_H$ will be the desired shadow. The contour of this shadow may be regarded as the section of a radial prism whose edges are light rays passing through the vertices of the given triangle.

To construct the shadow of a triangle on two projection planes, one has to apply the same sequence as recommended for construction of the shadow of a straight line (see Sec. 91). For instance, in Figs. 461

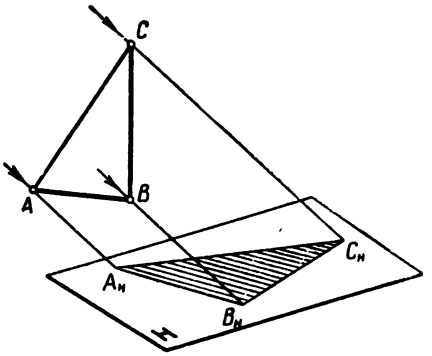

Fig. 460

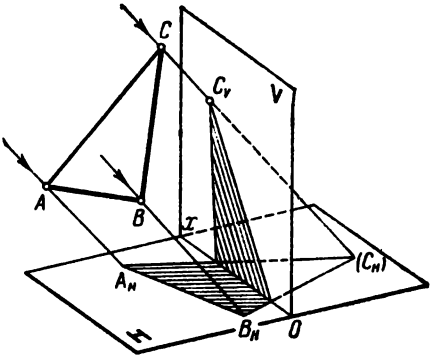

Fig. 461

and 462 the first to be built was the shadow of a triangle on the H plane on the assumption that there was no V plane.

The real shadow will be that which is located on the front half-plane H. This is the part of the shadow which we leave. Then we construct the shadow of the triangle on the V plane; to do this, it is

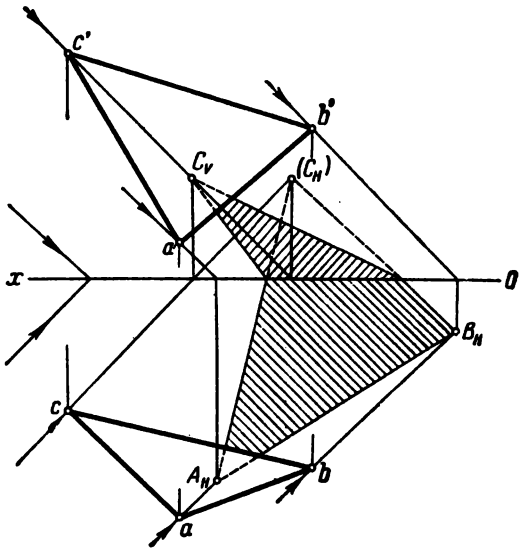

Fig. 462

sufficient to determine the shadow of the vertex C on the V plane. We complete the construction by joining C_V with the points of deflection of the shadows of sides AC and BC.

Through an investigation of the relative positions of the light rays and the plane of the given figure we determine the illumination of the projections of the figure. An instance of determining the shade of triangle ABC is given in Fig. 463. First, send a light ray DE through

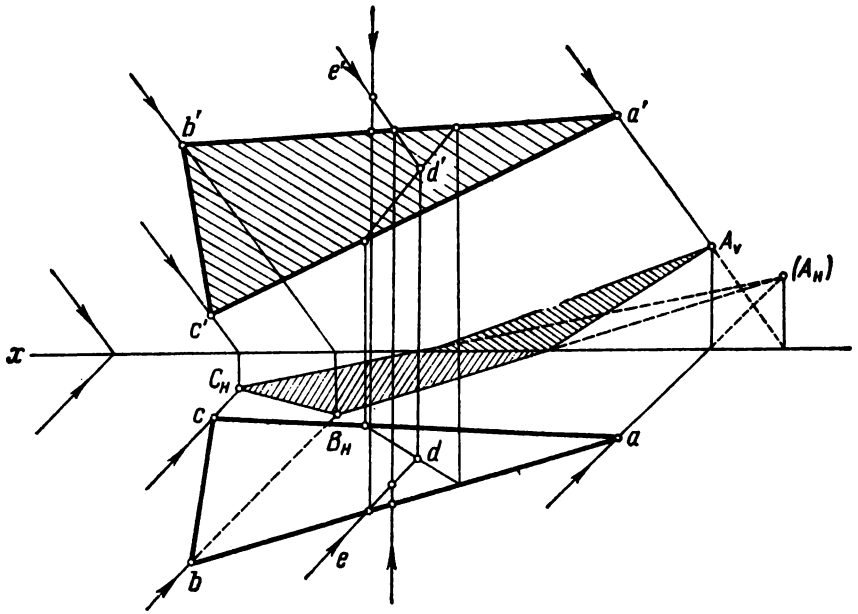

Fig. 463

point D lying within the contour of the triangle. Then establish the relative positions of the ray DE and the side AB as described in Sec. 13. The horizontal projection of a projecting ray directed perpendicular to the V plane and passing through the point of intersection of the vertical projections AB and DE shows that side AB is closer to the viewer than ray DE. Consequently, the side of the triangle that faces the viewer standing in front of the triangle and the V plane will be in the shade. That is why the vertical projection of the triangle in Fig. 463 is cross-hatched.

The projecting ray perpendicular to the H plane and passing through the intersection point of the horizontal projections AB and DE permits us to conclude that the horizontal projection (of the triangle) that is visible from above will be illuminated.

We consider another case of constructing the shadow of a triangle ABC when part of it lies beyond the limits of the first quarter (Fig. 464).

According to the procedure of Sec. 91, first isolate the visible part of the triangle (which is that part located in the first quarter). This requires finding the horizontal traces M and M_1 of sides AC and BC. Then construct the shadow of the triangle on one of the projection

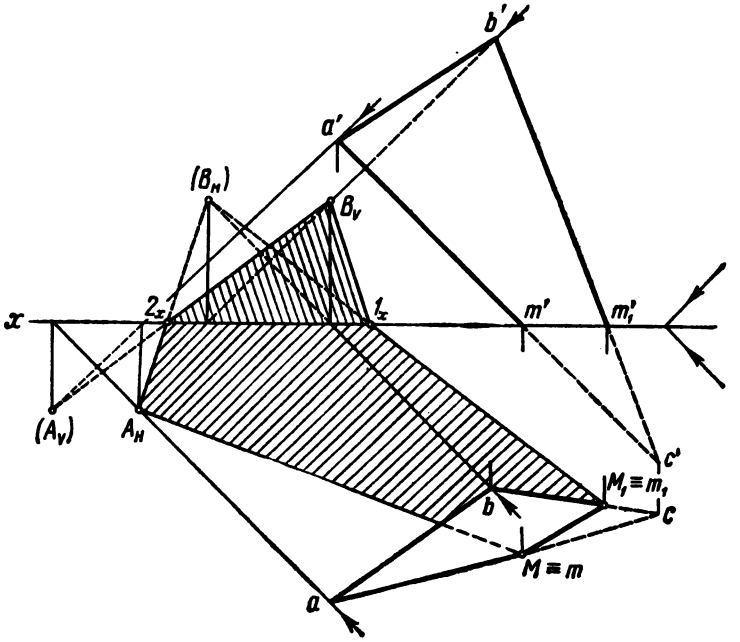

Fig. 464

planes on the assumption that the second does not exist; begin constructing the shadow on the plane which is intersected by the given figure.

In our case the triangle intersects the horizontal projection plane and so the first points to be found were A_H and (B_H): the horizontal traces of rays passing through the vertices A and B.

Connecting these points with M and M_1 respectively, we get the shadows of sides AC and BC on H. The contour of the shadow of the triangle is completed with the segment $A_H(B_H)$, which is the shadow of side AB. The intersection of the contour of the shadow with the coordinate axis Ox indicates that the shadow of the triangle will pass from the H plane to the V plane. Having determined the vertical trace of the ray (which passes through point B), we get B_V. Join the

shadow of point B on the V plane with the points of deflection 1_x and 2_x of the shadow. This yields the contour of the shadow of the triangle on the V plane, and completes the solution of the problem.

Sec. 93. The Method of Reverse Rays

The method of reverse rays is successfully employed in constructing shadows that are cast from one object to another.

First, construct the shadows of given geometric images on one of the projection planes and determine the points of intersection of the

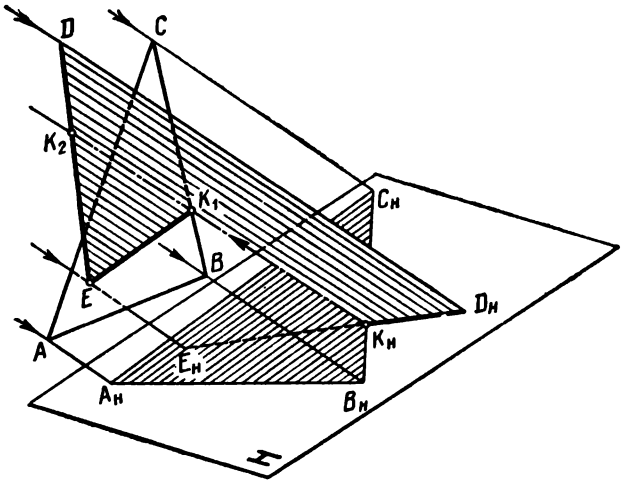

Fig. 465

shadows. Through the labelled points draw a ray in a direction counter to the light rays. Each reverse ray intersects the given geometric elements and determines the points that are needed to construct the shadow.

Let us see how this method is applied in the construction of the shadow of a straight line on the plane of a triangle. Fig. 465 shows the shadows of triangle ABC and line DE on the H plane. Through point K_H, which is common to the shadows of DE and side BC, draw a reverse (counter) ray intersecting these lines in points K_1 and K_2 respectively.

K_1 is the shadow of point K_2 of line DE on line BC. And the required shadow will be defined by the points K_1 and E, the latter being the intersection of line DE with the triangle.

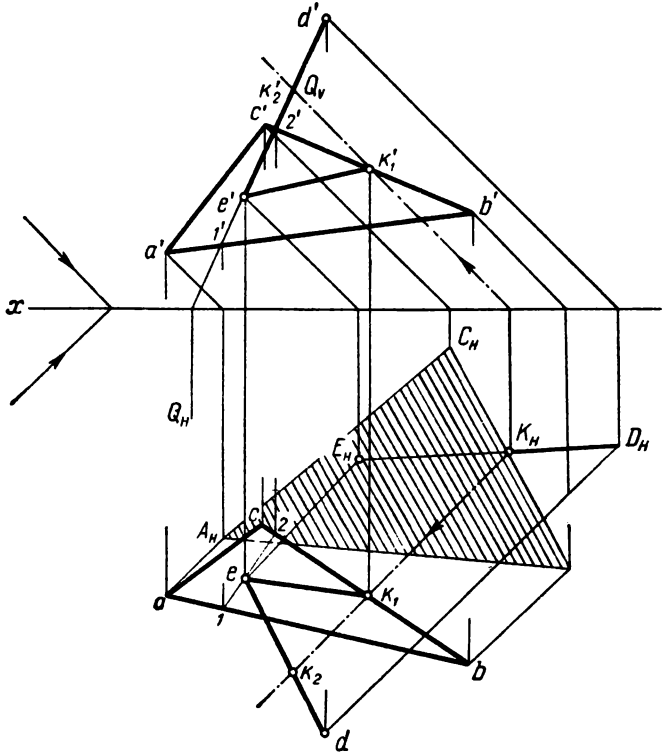

Fig. 466

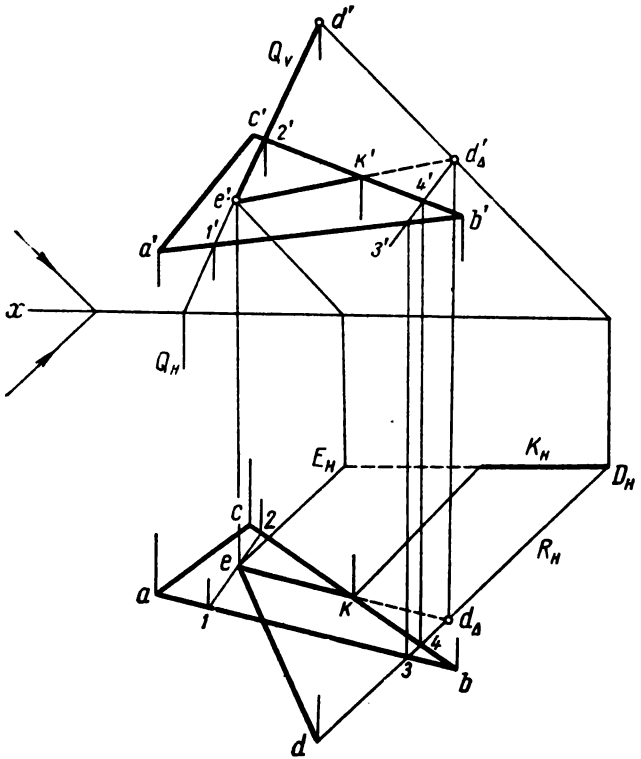

Fig. 467

This problem is solved orthographically in Figs. 466 and 467. In the first case, the shadow of line DE on the plane of the triangle is constructed by the reverse-ray method; in the latter case, with the aid of two points E and D_Δ, in which, respectively, the given straight line and the light ray passing through point D intersect the plane of the triangle. The planes Q (perpendicular to V) and R (perpendicular to H) are the projecting planes that are passed through the straight line DE and the ray to determine the indicated points. Since the point D_Δ is outside the contour ABC, part of the shadow of the straight line will be on the plane of the triangle and part will be on the projection plane.

A comparison of the two solutions enables us to conclude that in the former case we do not need to determine the point of intersection of the light ray (which passes through point D) with the plane of the triangle. The advantages of the reverse-ray method become rather obvious when constructing the shadows of a polyhedron on a polyhedron and when determining the shades of solids bounded by curved surfaces.

CHAPTER XVII

SHADES AND SHADOWS OF GEOMETRIC SOLIDS

Sec. 94. The Shades and Shadows of Polyhedrons

Let a certain polyhedron $SABC$ be illuminated by a beam of parallel rays (Fig. 468). It is required to construct the shadow and shade of this tetrahedron. To do this, draw through each vertex light rays parallel to the given direction and find their points of intersection with one of the projection planes, say H. In this way we

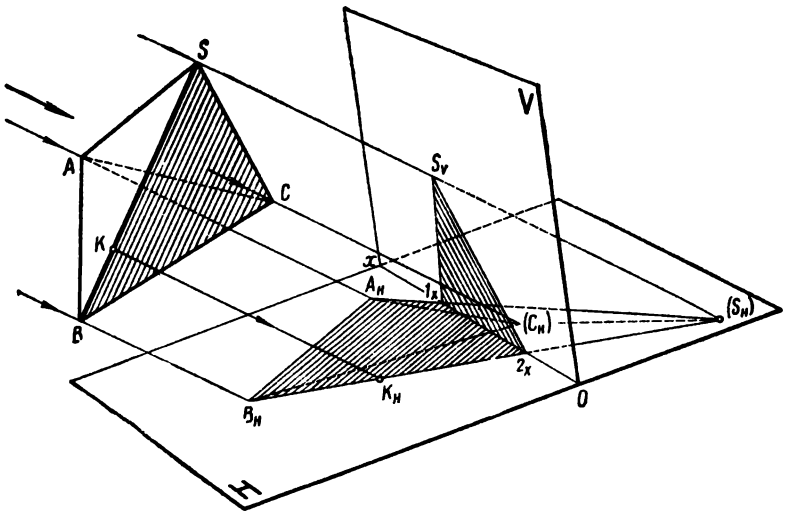

Fig. 468

get the shadows of the vertices of the tetrahedron on the H plane. Joining them, we obtain the shadow of the wire skeleton of the polyhedron. However, we are given not a skeleton but an opaque solid, the shadow of which should represent some kind of figure. In the case at hand, the contour of the shadow will be triangle $A_HB_H(S_H)$. This triangle is a section of the radial prism by the H plane. Since part of the shadow lies on the rear half-plane of H, it was necessary to determine also the shadow of the vertex S on the V plane. We join the

real shadow S_V and the points of deflection of the shadow on the x-axis.

The locus of points common to the surface of the radial prism and the given polyhedron forms a closed contour that separates the illuminated portion of the surface from the shaded portion. This is the so-called line of shade. To each point K of the line of shade there corresponds a point K_H on the line of shadow. Hence, the line of shadow

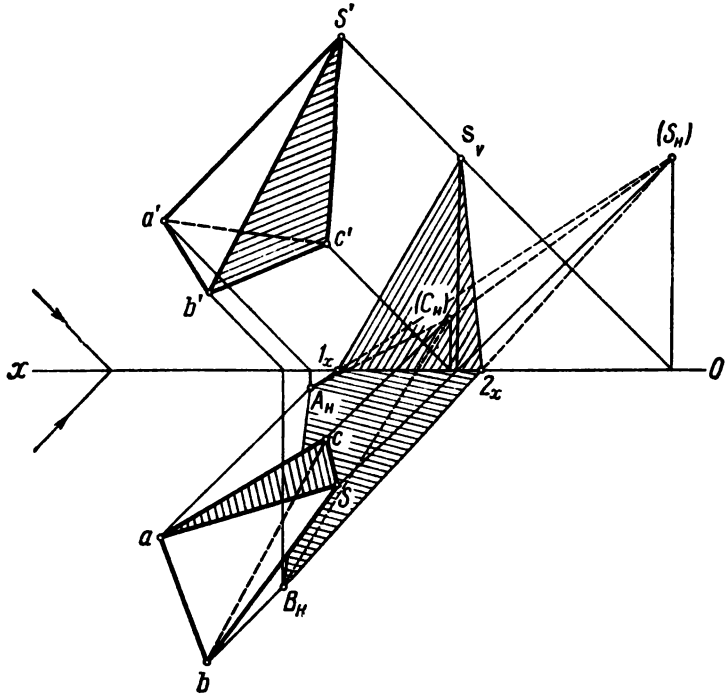

Fig. 469

is the shadow of the line of shade. We readily find the latter from the former. Indeed, in our case the line A_H-B_H-S_V-A_H limits the shadow, hence, the edges AB, BS, and SA will separate the illuminated faces of the tetrahedron from the shaded ones; that is, the closed line A-B-S-A will be the line of shade. This line limits the face ABS, which is illuminated because it faces the source of light. The remaining faces of the tetrahedron will lie in shade. Orthographically this problem is solved in Fig. 469, where we first construct the shadow of the tetrahedron on H and then, with the aid of the real shadow S_V, find the line of shadow on V.

The construction of shades and shadows is simplified if the base of the polyhedron is located on the projection plane. Then all vertices of the base will be coincident with their shadows. To construct the

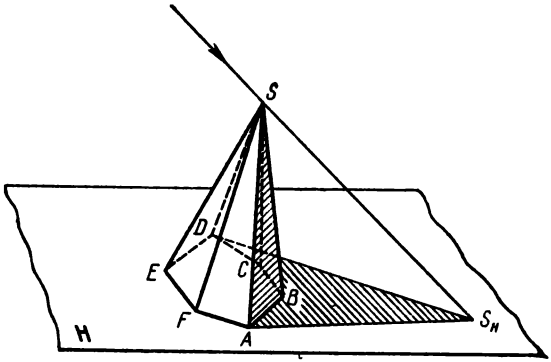

Fig. 470

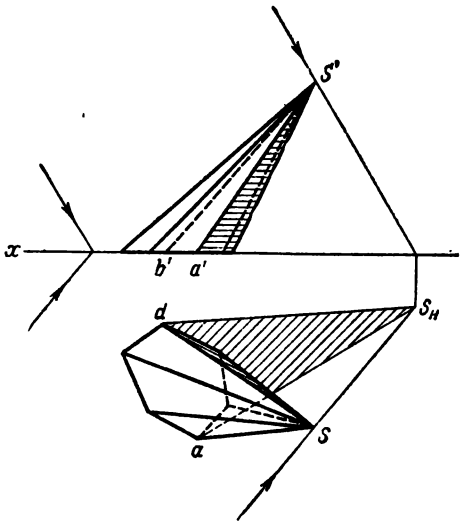

Fig. 471

shadow of a pyramid (Figs. 470 and 471), it is sufficient to find the shadow of its vertex and then join the point obtained, S_H, with the extreme points of the base. The faces of the pyramid that adjoin the shadow will lie in shade.

To determine the shadow of a prism on the plane of its lower base (Fig. 472), draw light rays through the vertices of the upper base and find the points A_{1H}, B_{1H},..., E_{1H} of intersection of the rays with H.

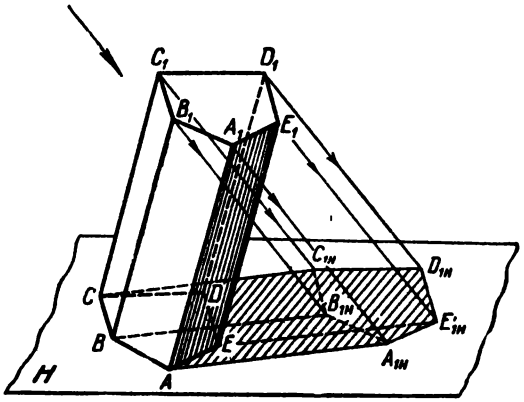

Fig. 472

The figure $AA_{1H}E_{1H}D_{1H}E_{1H}C$ will be the required shadow of the prism on H. From the line of shadow we find the line of shade A-A_1-E_1-D_1-C_1-C-B-A.

Sec. 95. The Shadows of Intersecting Polyhedrons

The problem of constructing the shadows of intersecting polyhedrons is not confined to determining the lines of shade and shadow on the projection plane. A solution is completed by constructing the shadows cast by the shaded faces of one solid on the illuminated faces of another that intersect these shaded faces.

Each line of the desired contour will be an intersection of the radial plane passing through the edge of a shaded face of one polyhedron with the illuminated face of the other. Thus, underlying all constructions is a determination of the shadow of a straight line on a plane. This problem was considered in Sec. 93 where we resorted to the method of reverse rays. The essence of this method is illustrated in Fig. 473, which gives the construction of the shadow of a straight line AB on the face SEF of a triangular pyramid. It is assumed that the shadows of the straight line and the pyramid on H have already been found. From the intersection points of the shadows (points *1* and *2*) draw rays counter to the light rays. Points K_1 and K_2, in which the counter (reverse) rays meet the edges of the pyramid, determine the straight line which is the shadow of AB on the face SEF. Fig. 474 illus-

trates the use of the reverse-ray method in an orthographic solution of the problem.

Now let us examine two intersecting polyhedrons shown in Fig. 475. First construct the lines of shadows on the H plane; these yield the shades. In Fig. 475, the two lateral faces SDE and SDF of the pyramid and two (ABB_1A_1 and BCC_1B_1) of the prism are shaded. Now we have to find out whether there are cases when a shaded face of one solid intersects an illuminated face of the other.

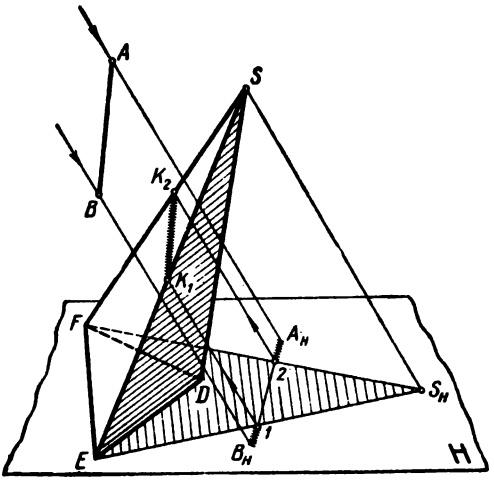

Fig. 473

In our example, the two shaded faces of the prism intersect the illuminated face SEF of the pyramid, and the shaded face of the pyramid SDE intersects the illuminated face ACC_1A_1 of the prism. Let us examine each pair of faces in succession.

In the construction of the shadow from face ABB_1A_1 of the prism on face SEF of the pyramid we employ reverse rays drawn through points *1* and *2*. In these points, the shadows of edges SE and SF intersect the shadow of edge A_1B_1. Points K_1 and K_2 obtained with the aid of reverse rays determine the shadow of edge A_1B_1 on the face of the pyramid. The shadow from the pyramid on face ACC_1A_1 of the prism will be limited by the shadow of edge SE. The shadow of this edge, deflected at point M, will pass from the H plane onto the illuminated edge of the prism. Point I (in which edge SE intersects the face) and point M determine the desired boundrary of the shadow. The line of shadow from face BCC_1B_1 on the illuminated

face of the pyramid will be invisible and therefore we do not construct it.

The method of reverse rays is extremely convenient, but it is not the only method for constructing the shadows of a polyhedron on a

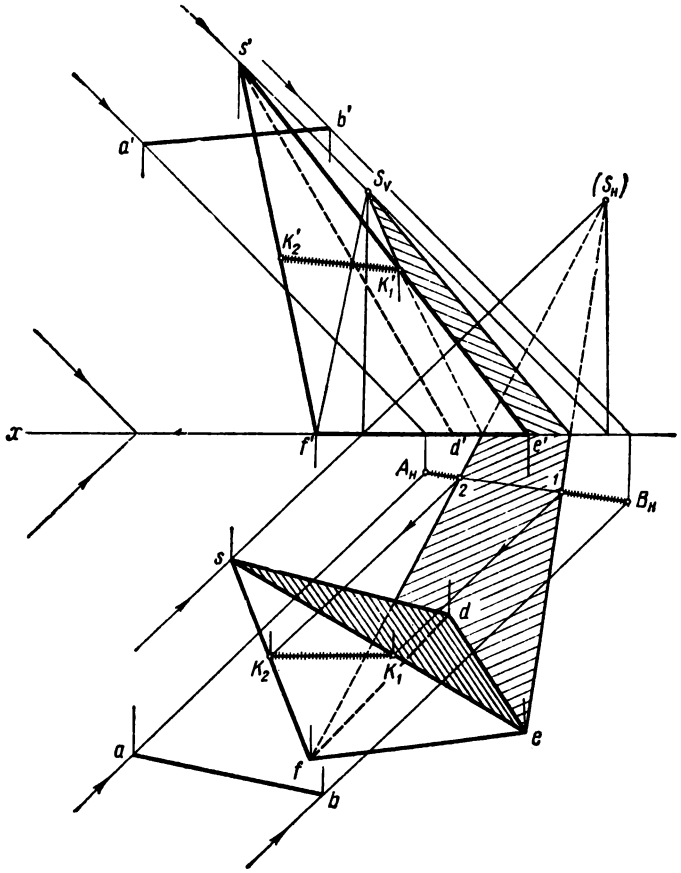

Fig. 474

polyhedron. In some cases it is advisable to utilize the points of intersection of edges and faces on which the shadow of the given edge falls. These points may not always be within the limits of the face. For instance in Fig. 476, the shadow of the edge SA of the pyramid on a face of the prism is constructed with the help of points K_1 and

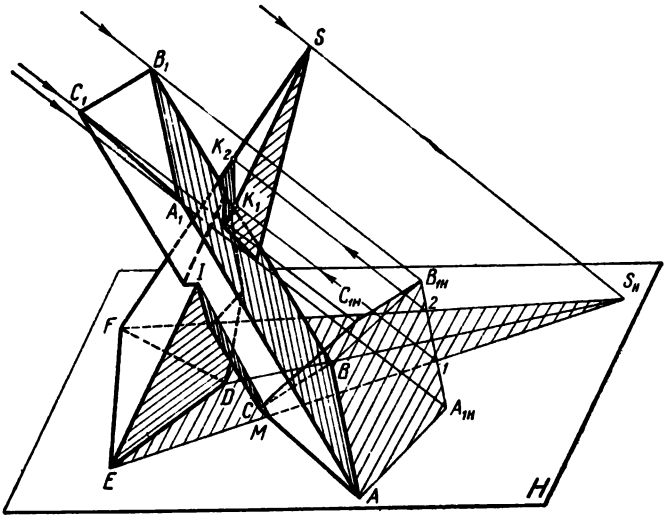

Fig. 475

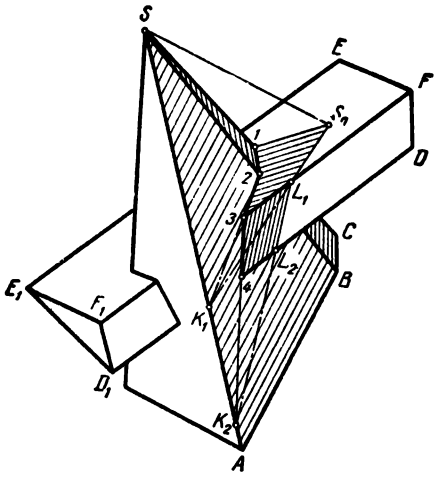

Fig. 476

K_2, in which edge SA intersects the extended (beyond their limits) faces EFF_1E_1 and FDD_1F_1 of the prism. The shadow of edge SA on the plane of the first face is defined by points K_1 and S_n. The latter is the shadow of vertex S on face EFF_1E_1. In point L_1 the shadow of the edge is deflected and passes from one face to another.

Sec. 96. The Shadows of Curved Surfaces

In the general case, light rays touching a solid of arbitrary shape form a cylindrical surface and not a prism, as was the case with polyhedrons. The line of tangency of a radial cylinder and a given solid separates the illuminated portion from the shaded portion. This is the line of shade. The line of intersection of a radial cylindrical

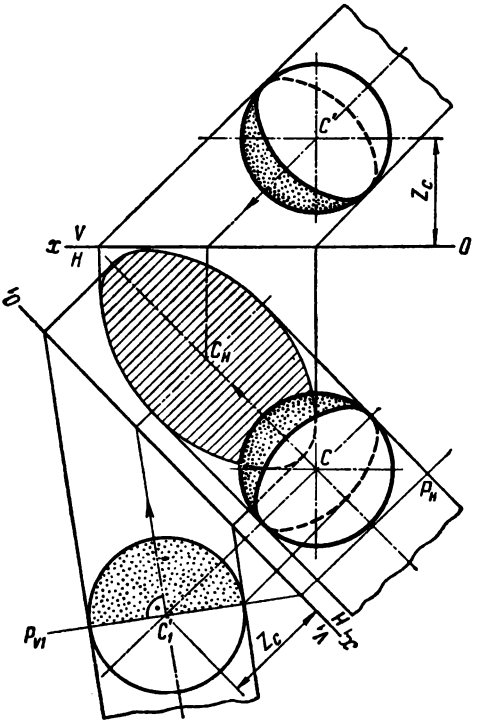

Fig. 477

surface with the projection plane or some other solid is the line of shadow. As before, the line of shadow is the shadow of the line of shade.

In some cases each line may be constructed independently; in others it is best first to determine the line of shade and then from it the line of shadow; in still other cases it is advisable to construct in the reverse order. The examples which follow will serve to illustrate these statements.

Example 1. *To construct the shade and shadow of a sphere* (Fig. 477). In this case, the radial cylindrical surface will be circular in sections

perpendicular to the generatrices. One of these sections by plane P passing through the centre of the sphere will be the line of shade. We omit describing the construction of the projections of the section and note that the new plane V_1 is parallel to the light ray. In the new V_1/H system, plane P has become a projecting plane. The trace of the radial surface serves as the line of shadow. Here it will be an ellipse, the centre of which is the shadow of the centre of the sphere.

Example 2. *To construct the shades and shadows of cones and cylinders occupying different positions in space.*

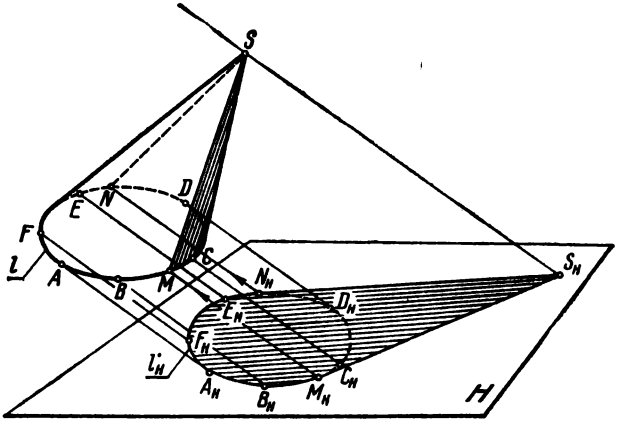

Fig. 478

The light rays tangent to a cone form two planes and a cylindrical surface (Fig. 478). To construct the line of shadow, determine the shadows of the vertex and the base of the cone. Then from point S_H draw lines tangent to the curvilinear contour l_H, which is the shadow of the cone base. Segments of tangent lines $S_H M_H$ and $S_H N_H$ are shadows of the generatrices of the cone which separate the illuminated portion from the shaded one. Using reverse rays, we get points M and N on the base of the cone; we complete the solution by connecting these with the vertex S. Bear in mind that the generatrices thus found and the arcs of the base (MAN) will be the line of shade.

Construction of the shade and shadow of a cylinder (Fig. 479) differs in only one respect: in place of the shadow of the vertex S we have to determine the shadow of the second (upper) base.

Fig. 480 illustrates an orthographic solution of this problem in which we first construct the shadow of the cylinder on the H plane. The tangent lines to the two ellipses l_H and l_{1H} are the shadows of the generatrices AA_1 and BB_1 that separate the illuminated portion of

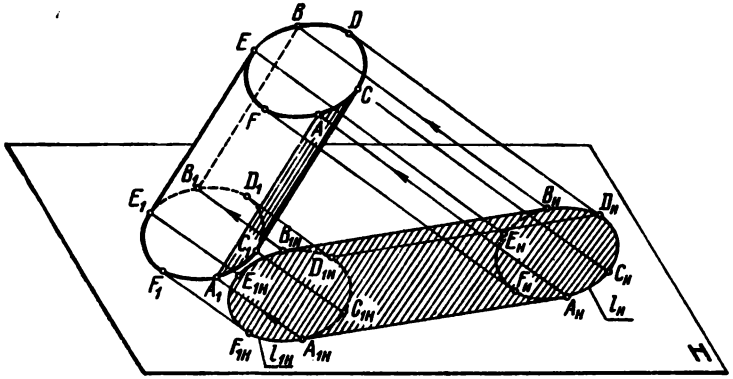

Fig. 479

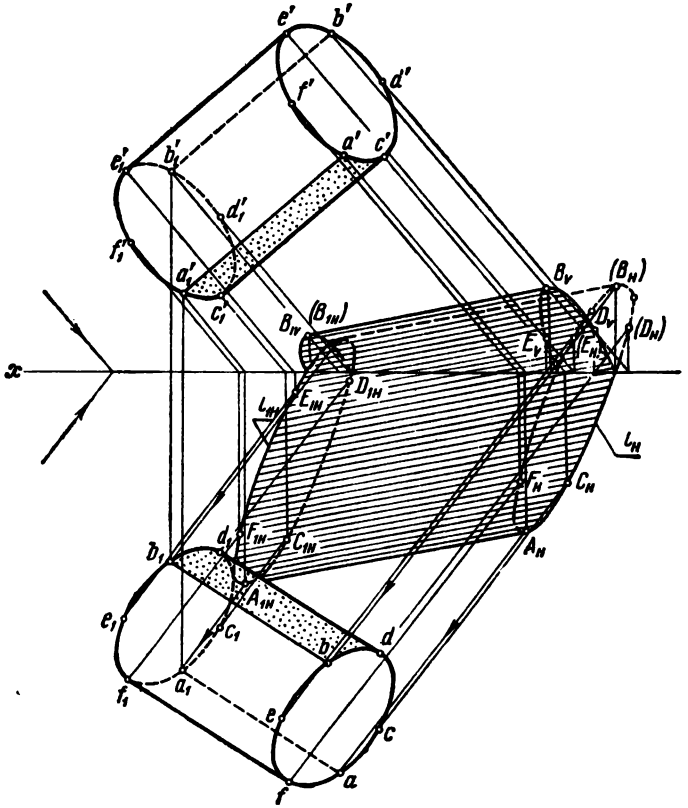

Fig. 480

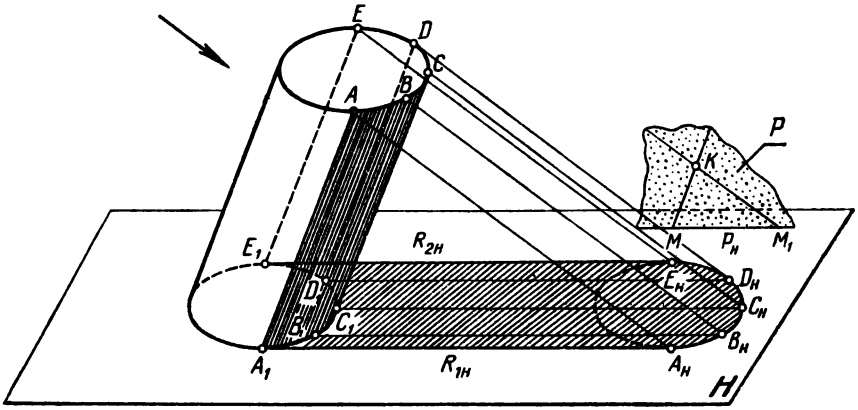

Fig. 481

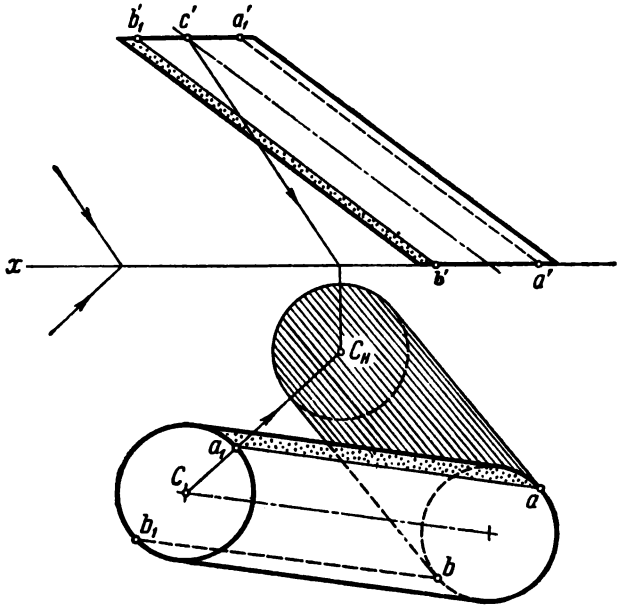

Fig. 482

the cylinder surface from the shaded portion. Points A, A_1, B and B_1 on the bases of the cylinder are obtained by reverse rays. Since part of the shadow is on the rear half-plane of H, in order to complete the construction we had to determine the vertical traces of the rays passing through points B, D, E and B_1.

The solution is appreciably simplified if the base of the cylinder or the cone is located in one of the projection planes. In this case, we do not need to find the shadows of points of the base lying in the projection plane.

The rectilinear outline of the shade and shadow of a cylinder (Fig. 481) is readily obtained by means of planes R_1 and R_2 that are tangent to the cylinder and are parallel to the light ray. To determine the direction of the traces R_{1H} and R_{2H} it is necessary to construct plane P parallel to the light rays and the generatrices of the cylinder (see

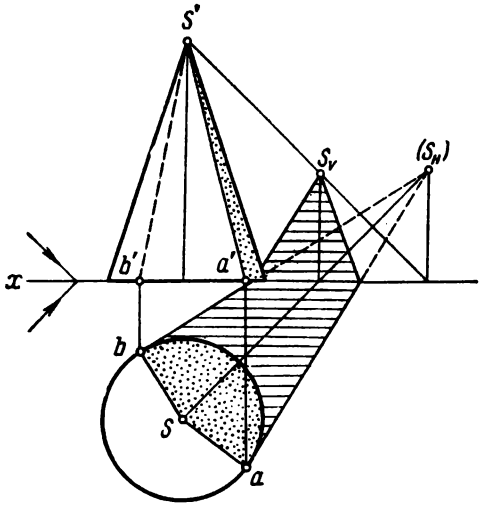

Fig. 483

Sec. 48, Example 3). The generatrices A_1A and E_1E, along which the planes R_1 and R_2 touch the cylinder will separate the illuminated from the shaded portion. Arc ACE of the upper base completes the line of shadow.

The example in Fig. 482 is peculiar in that, here, only one light ray was used to construct the shade and shadow of a cylinder; with this light ray we find the shadow C_H of the centre of the upper base. Since the upper base of the cylinder and its shadow lie in two parallel planes, they are two congruent figures (in our case, two equal circles). In constructing the shade and shadow of the cone shown in Fig. 483, the radial surface is converted into two planes tangent to the cone and parallel to the light rays

The traces of these planes limit the shadow of the cone, and the generatrices SA and SB, along which the planes are tangent to the cone, are lines of shade.

Example 3. *To construct the shade and shadow of a general double-curved surface of revolution* (Fig. 484).

Let us first consider constructing the desired lines (of shade and shadow) for the case when the light rays are parallel to the V plane.

The curve along which the given surface is tangent to the radial cylinder will be the line of shade. Each of the light rays touches the

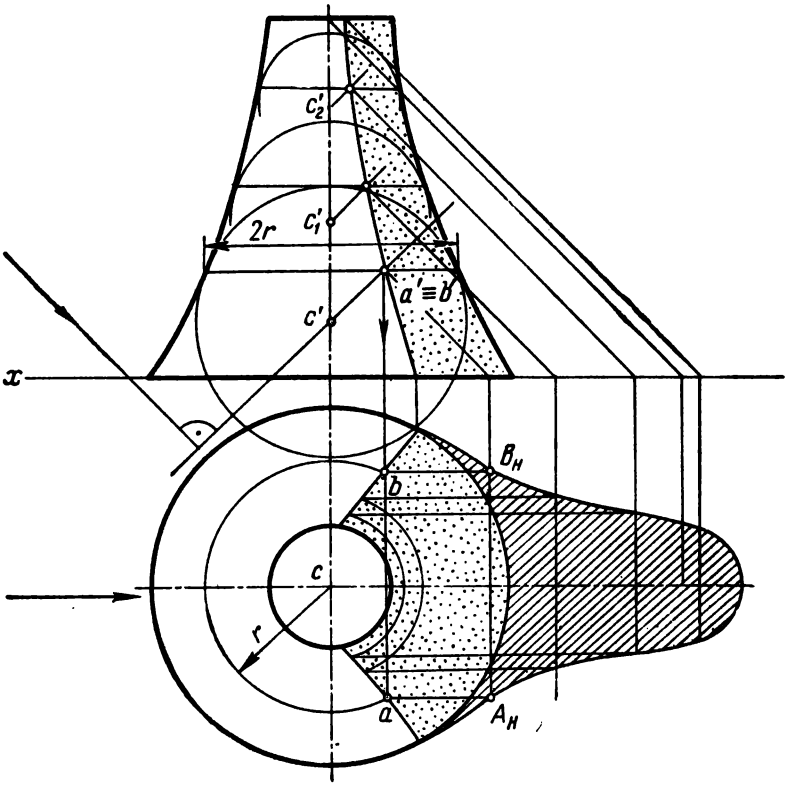

Fig. 484

surface of revolution in some ordinary point A and must belong to the plane tangent to the surface passing through this point. The problem may thus be reduced to determining the locus of points in which a given surface touches planes parallel to the light ray. To solve the problem thus formulated we inscribe spheres in the surface of revolution and construct the projections of the circles along which each sphere is tangent to the given surface. The sphere with centre in point C is tangent to the surface of revolution along a circle of radius r.

The radius of the auxiliary sphere drawn to the desired point of tangency must be normal to the tangent plane. Hence, the vertical projection of the radius $c'a'$ must make a right angle with the like projection of the vertical line of the tangent plane. In our case, the tangent plane must be parallel to the vertically located light rays. That is why, in Fig. 484, $c'a'$ is perpendicular to the vertical projection of the ray. Point A, in which the radius intersects the circle of tangency of the sphere and the surface of revolution, will belong to

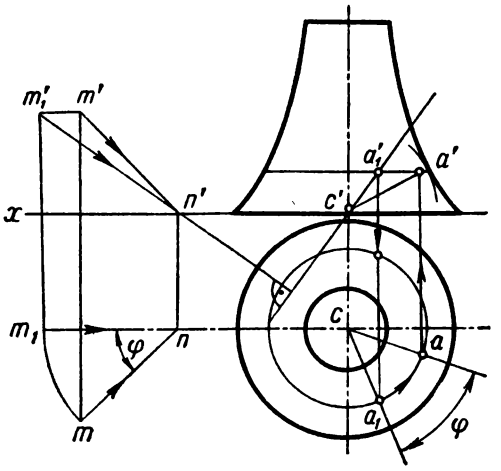

Fig. 485

the required line of shade. The horizontal projection of point A and of B (which is symmetric to it) is determined as the point of intersection of the horizontal projection of the parallel of the radius r and the line of recall. The remaining points are found in similar fashion. The line of shadow is constructed as the shadow of the line of shade.

The supplementary constructions that have to be executed if the light rays are not parallel to the projection planes are shown in Fig. 485. Here, we first construct new projections (m_1n and $m_1'n'$) of a light ray revolved through an angle φ to a position parallel to V. Then, in accord with the earlier described procedure, we find the projection of point A_1 that lies on the line of shade. It now remains to transform the orthographic representation back again: revolve the points about the axis of the surface of revolution through an angle φ counterclockwise.

Sec. 97. Shades and Shadows on the Facades of Buildings

The construction of shades and shadows on the facades of buildings is based on determining the points of intersection of light rays with the vertical planes of the facade or with the inclined slopes of the roofs. Besides the facade of a building, constructions require another given projection (the plan or a side view).

When determining the line of shadow, which is a parallel projection of the line of shade, it is wise to take advantage of the following rules:

1. The shadow of a plane figure on a parallel plane is congruent to that figure, because the lines of the section of the radial cylinder by parallel planes are always congruent. For one thing, the shadow of a line-segment on a parallel plane is equal and parallel to the segment itself.

2. If a straight line is perpendicular to a certain plane P, the shadow of the line on P coincides with the direction of the projection of the ray onto this plane. This is due to the fact that the light rays here form a plane which will be a projecting plane with respect to P.

Consequently, if a line is perpendicular to V, which means also to the plane of the principal facade, the radial plane passing through this line will be the vertical projecting plane. The vertical trace of the radial plane will be parallel to the like projection of a ray. Such a plane will intersect any surface of the facade of a building along a line, the projection of which on V is a straight line coincident with the trace of the plane.

Let us see how these rules are applied in constructing the shadows on the facade of a building, the orthogonal projections (facade and plan) of which are shown in Fig. 486. Here, line-segment KL specifies the direction of a light ray.

For this purpose let us consider a determination of the lines of shadows on fragments of a given facade that are depicted on a larger scale.

Fig. 487 gives the construction of the shadow of a cornice and window niche.

On the edge of the cornice I-I take a point A $(a,\ a')$, and through it draw a ray and find point A_1 $(a_1,\ a_1')$ where it meets the vertical plane of the lower projection. The shadow cast by line I-I will pass through A_1 parallel to the line itself. In the same way, using a ray that passes through point B, we find the shadow from the edge II-II on the plane of the wall.

Then determine the shadow of line CD. Since this straight line is perpendicular to the plane of the facade, the shadow it casts will be parallel to the vertical projection of the ray. Finding the points of

intersection of the light rays drawn through the angles of the edges of the cornice (through points C, I and II) with the plane of the central part of the facade, we complete the construction of the line of shadow of this cornice. It will be noted that the horizontal lines drawn through $1'_1$ and $2'_1$ are respectively the shadows of edges I-I and II-II on the plane of the central wall.

Now let us define the shadow in the niche of the window. The line of shadow of the niche will be a figure congruent to the outline of

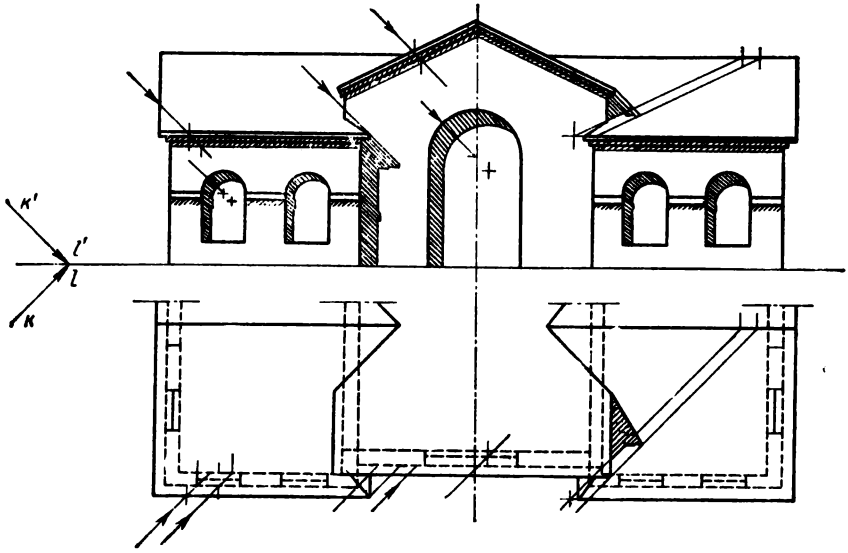

Fig. 486

the niche, but displaced a distance ΔS in the direction of the vertical projection of the ray. We find this displacement by determining the shadow of the centre of the arc of the circle (point Q_1). To this line we have to add the shadow cast by the projection of the horizontal belt (this is done with the aid of rays passing through points E and F).

The shadow from the cornice of the pediment is constructed in Fig. 488, where the lateral facade (profile projection) is taken as a secondary projection. Without repeating the description for determining the shadows of the edges of the cornice on the vertical planes of the projections, we note certain peculiarities in construction of the shadow of the right angle of the cornice on the inclined slope of the roof. The shadow of line AB, perpendicular to V along the plane of the slope, will be directed parallel to the vertical projection of the ray and will be limited by the shadow of point A. The latter is defined as

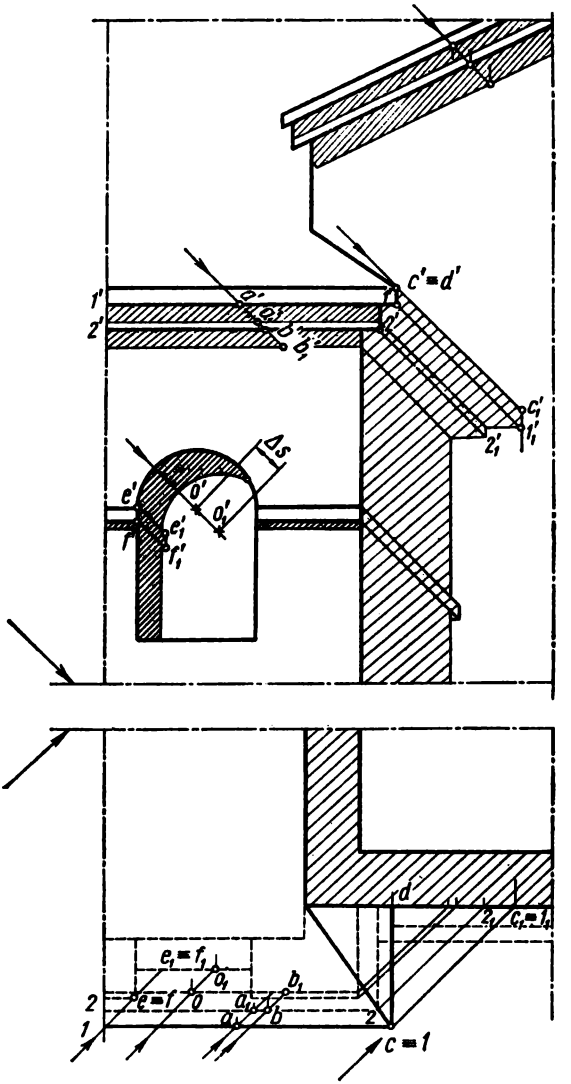

Fig. 487

the point of intersection of the ray with the profile projecting plane of the roof slope; that is to say, we first find its profile projection a_1'', and then the vertical projection a_1. The other points of the desired line are obtained in the same way.

In conclusion we consider an example of determining the shadow cast by a smokestack on the plane of a roof (Fig. 489). We omit the description of constructions to find points of intersection of light rays with the profile projecting plane of the roof slope and shall prove that the

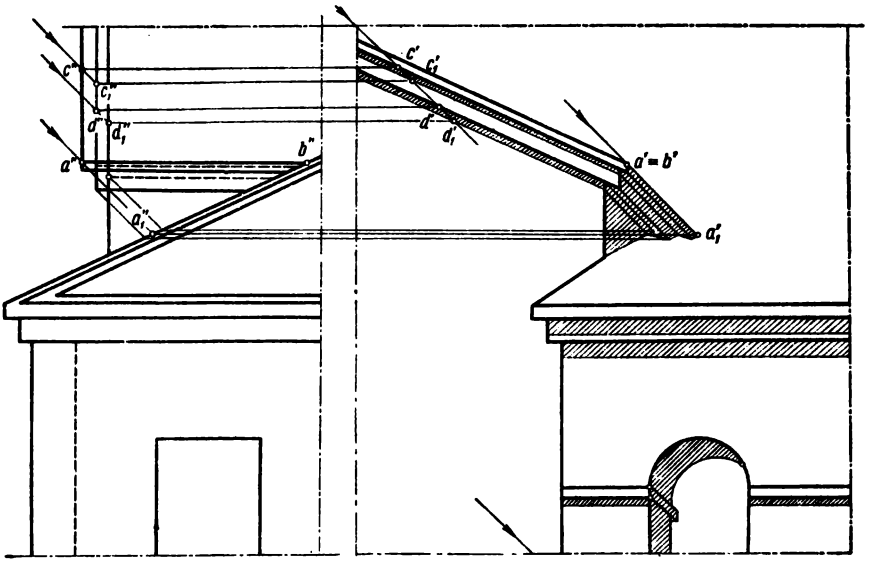

Fig. 488

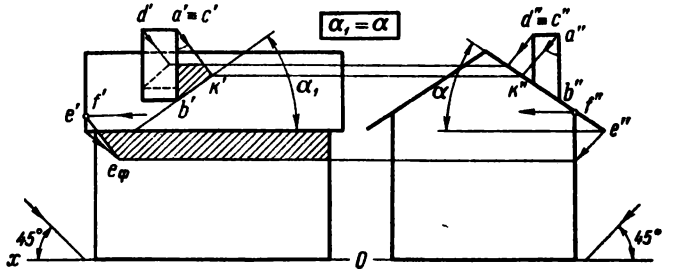

Fig. 489

shadow from the vertical edges of the stack on the inclined plane of the roof is projected on V into a straight line making with the x-axis the same angle α that is formed with H by the given roof slope.

Indeed, from the equality of triangles $a'b'k'$ and $a''b''k''$ (two equal sides and the included angle) it follows that angle $a'b'\kappa' =$ $=$ angle $a''b''k''$ and, hence, $\alpha_1 = \alpha$ as complementary equal angles.

CHAPTER XVIII

SHADES AND SHADOWS IN PERSPECTIVE AND AXONOMETRY

Shades and shadows of objects are often added to give expression to perspective and axonometric projections. Such constructions are based on the principles considered in Chapters XVI and XVII devoted to the geometric theory of shades and shadows and its applications in orthogonal projections.

Without repeating them, let us examine certain concrete cases of the construction of shades and shadows in perspective and axonometry. They will serve as illustrations of the peculiarities of these methods.

Sec. 98. Shades and Shadows in Linear Perspective

Since the light source S is considered to be at infinity, the secondary projection of it must be on the line of the horizon. Now the perspective of a light source (the sun) may be located either below the

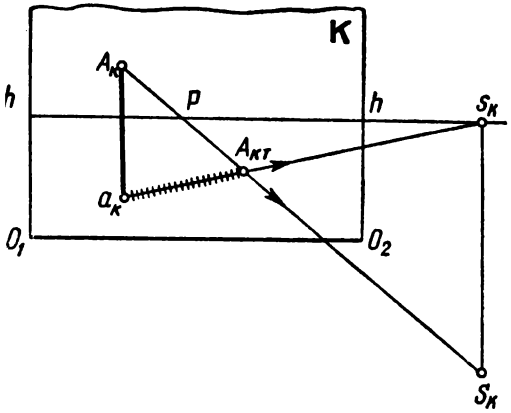

Fig. 490

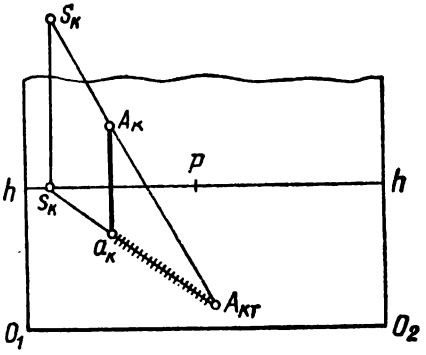

Fig. 491

horizon or above it; in both cases the sun is situated above the plane of the horizon H.

In the first case (Fig. 490), the source of light is located in imaginary space (behind the viewer), in the second instance (Fig. 491), it lies in object space (in front of the viewer but behind the object).

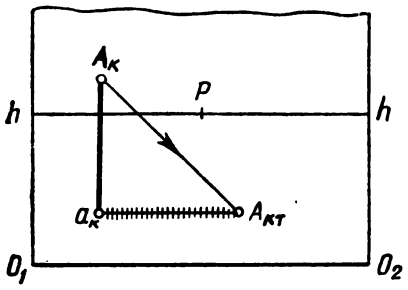

Fig. 492

These figures illustrate the construction of the shadows of points and vertical line-segments. To obtain in perspective the shadow A_{KT} of point A_K on the ground plane T, through the given point draw the perspective of a ray (line $S_K A_K$), and through its secondary projection a_K, the secondary projection of the ray (straight line $s_K a_K$). The intersection of the perspective of the ray with its secondary projection will define the shadow A_{KT} of point A_K cast on plane T. The shadows of the vertical segments will coincide with the direction of the secondary projections of light rays passing through points a_K.

The direction of light rays is most often taken parallel to the picture plane (Fig. 492). Then the secondary projections of the rays must be parallel to the ground line of the picture, that is, horizontal.

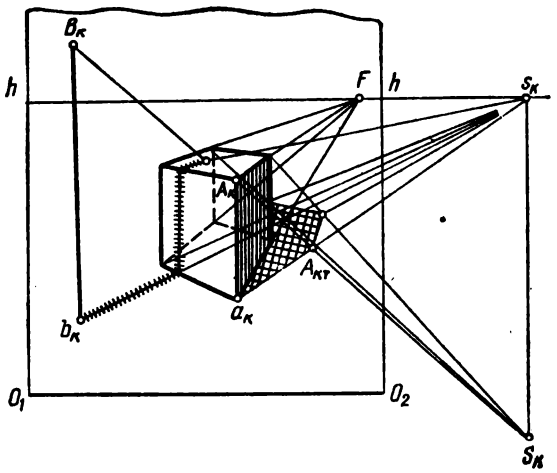

Fig. 493

In Figs. 493-495 the shades and shadows are constructed of a rectangular parallelepiped for three different positions of the light source.

Less appropriate is the position of point S_K, where the vertical planes of the visible faces of the parallelepiped are in the shade and the shadow faces the viewer in the form of an excrescence (see

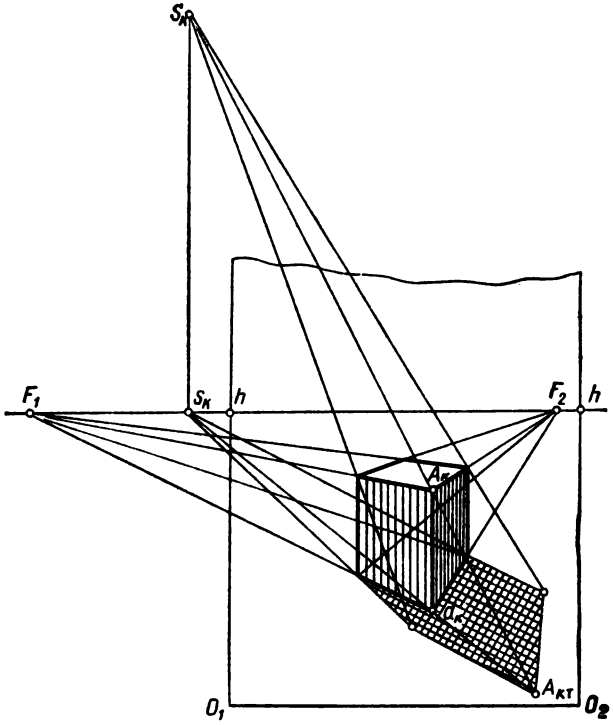

Fig. 494

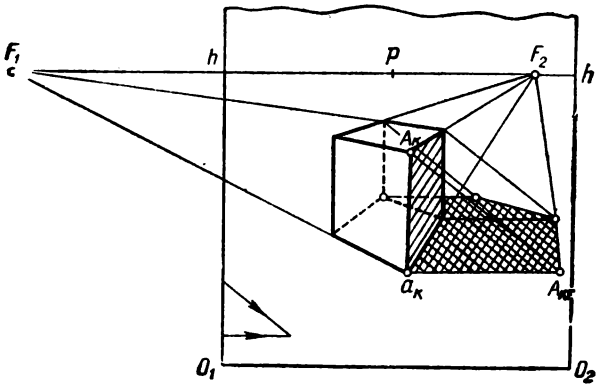

Fig. 495

Fig. 494). There are obvious conveniences in the construction where the light rays are parallel to the picture plane (see Fig. 495). Note that then not only the secondary projections but also the light rays themselves will be parallel.

The first of these three examples (see Fig. 493) illustrates the construction of a shadow cast by a vertical pole onto a vertical face and horizontal face of a parallelepiped. On the former face the shadow is parallel to the line-segment itself. In the horizontal face it is directed towards the vanishing point of secondary projections of the rays (to point s_K, which at the same time is the secondary projection of the light source S).

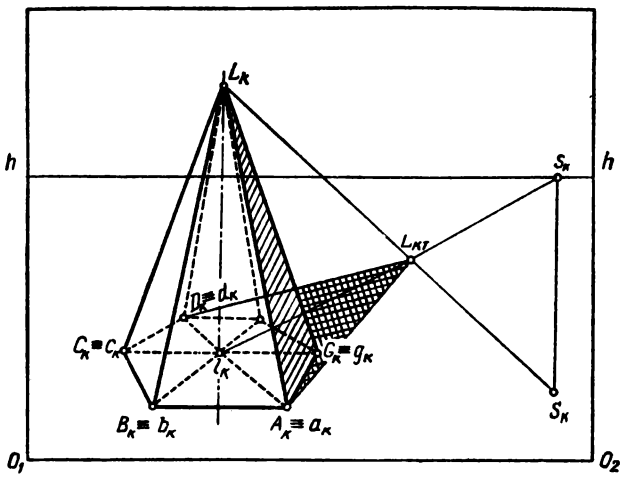

Fig. 496

Figs. 496 and 497 give the construction of the shadows of a pyramid and a cone. In both cases we first find the shadows of the vertices (points L_{KT}), from which we then draw lines $L_{KT}A_K$ and $L_{KT}D_K$. In Fig. 496 these will be the shadows of the edges, in Fig. 497 the shadows of the generatrices that define the lines of shade of the geometric solids.

Fig. 498 is an instance of the construction of shades and shadows of a right cylinder whose base lies in plane T.

The rectilinear line of shadow of the cylinder will be the traces of radial planes tangent to the cylinder. Lines tangent to the base of the cylinder drawn from point s_K (the secondary projection of the light source S) serve as the traces of these planes in Fig. 498.

The points of tangency a_K and b_K define the generatrices A_Ka_K and B_Kb_K that belong to the line of shade. This line is completed by

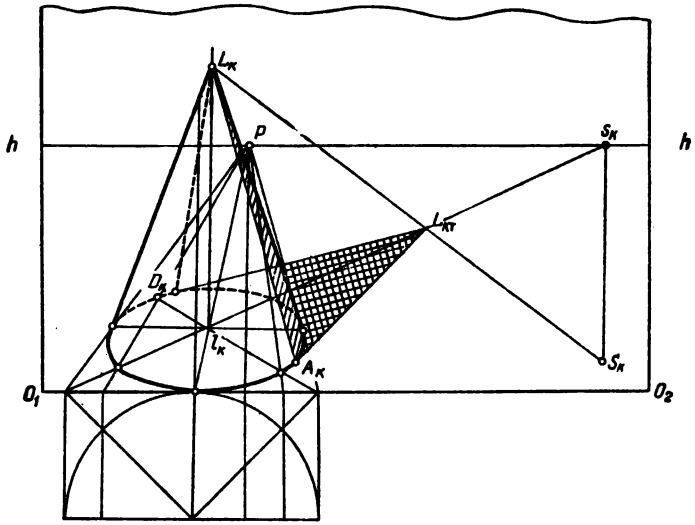

Fig. 497

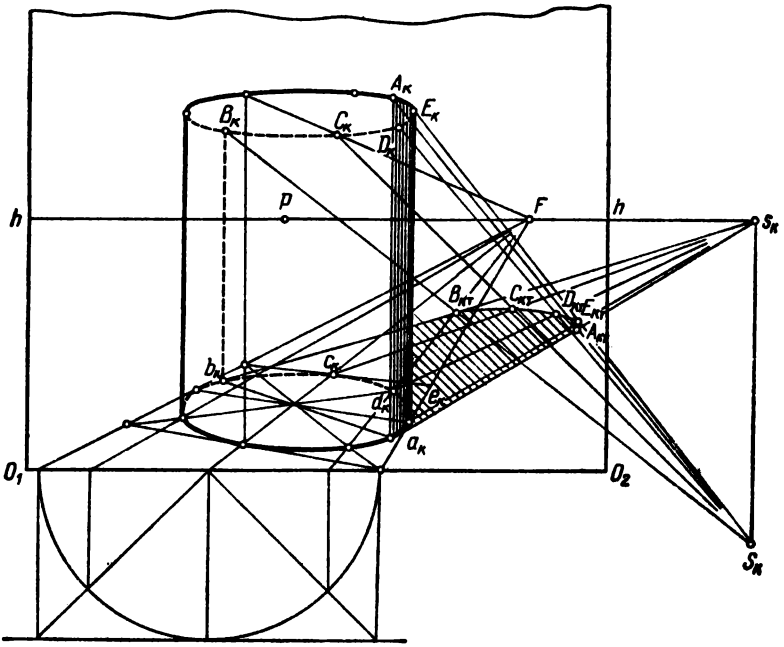

Fig. 498

the arc $A_K D_K B_K$ of the upper base. We complete construction of the shadow by finding the shadows of points of this arc on the ground plane.

In conclusion we consider two examples of the construction of shadows cast by one object onto another.

Fig. 499 shows the construction of a shadow from a vertical wall cast onto a stairway. The light rays are parallel to the picture.

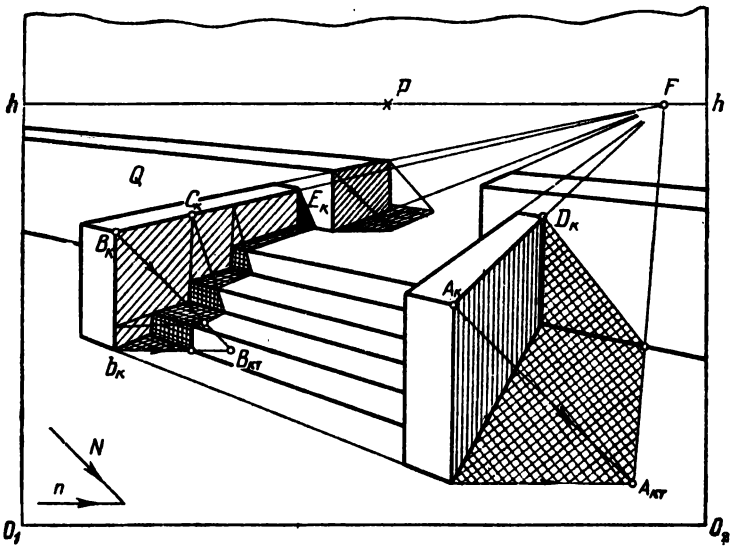

Fig. 499

1. The shadow from the vertical edge $B_K b_K$ on the ground plane and on the horizontal plane of the step will be parallel to the secondary projection of the ray, that is, parallel to the ground line.
2. The shadow from that same edge $B_K b_K$ on the vertical planes of the risers, parallel to it, will be parallel to the edge itself.
3. The shadow from the horizontal edge $B_K E_K$ on the planes of the steps parallel to it has a common vanishing point with the edge.
4. The shadow from edge $B_K E_K$ on the planes of the risers will be directed to points in which $B_K E_K$ intersects the planes of the risers extended upwards. One of these points (C_K) is shown in Fig. 499.

In the same way, the shadows of edges $B_K E_K$ and $A_K D_K$ on the vertical plane Q will pass through the points of intersection of these edges with the plane Q.

Now turn to Fig. 500, which portrays the perspective of roof slope $L_K M_K N_K R_K$ and the edge of stack $A_K B_K$.

To construct the shadow of this edge on an inclined roof slope, pass through a light ray an auxiliary vertical plane Q, the trace of which on the ground plane coincides with the secondary projection of the ray (straight line 1_K-2_K). Then construct the line of intersection of plane Q and the given roof slope. The perspective of this line is labelled I_K-II_K and the secondary projection is labelled 1_K-2_K.

Point A_{KP}, in which the ray intersects line I_K-II_K, is the shadow of point A on the inclined plane of the roof slope. Connecting A_{KP} with the point of intersection of edge and roof (with point B_K), we get the required shadow of the edge.

Fig. 501 illustrates the principle of determining the shadow cast by the roof ridge onto the plane of another slope and the smokestack. Here the constructions depicted in Fig. 500 were executed several times.

Sec. 99. Shades and Shadows in Axonometry

Determination of the lines of shadow in axonometry is essentially the same as in the case of constructing shadows in perspective when the light rays are parallel to the picture plane.

But in axonometry the shadow of any straight line on a plane parallel to the line will be parallel to the line itself. In perspective this occurred only in particular cases when the straight line and the plane were vertical or parallel to the ground line.

It is advisable to determine the direction of the light ray by the diagonal of the cube, the faces of which are coincident with the coordinate planes and the edges are constructed with allowance made for foreshortening (distortion factor) (Fig. 502).

The diagonals of the cube faces will be secondary projections of the ray on the coordinate planes.

Consequently, the shadow of a line perpendicular to a coordinate plane will be parallel to the secondary projection of the ray on this same plane.

Thus, shadow $a_K A_{KH}$ of vertical edge $A_K a_K$ is directed along the secondary projection of the ray on plane $x_K O_K y_K$ (along the diagonal of the lower face); the shadow of edge $A_K C_K$ ($A_K C_K$ is perpendicular to V) is defined by points A_{KV} and C_K, which means that it coincides with the secondary projection of the ray on the plane $x_K O_K z_K$.

On the same drawing we have the shadow $A_{KH} C_{KH}$ of edge $A_K C_K$ on the horizontal plane.

We again stress the fact that since edge AC is parallel to plane xOy, the shadow of the edge on this plane and the edge itself will be parallel. The foregoing peculiarities must be kept in mind when solving concrete problems.

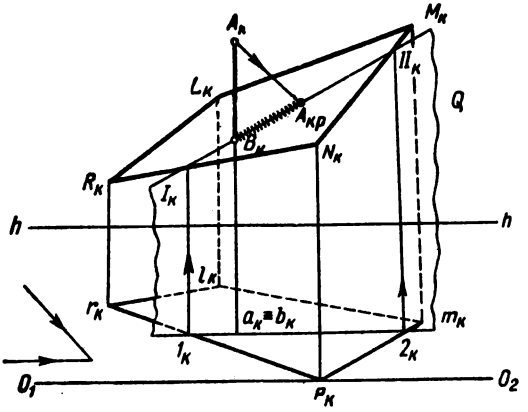

Fig. 500

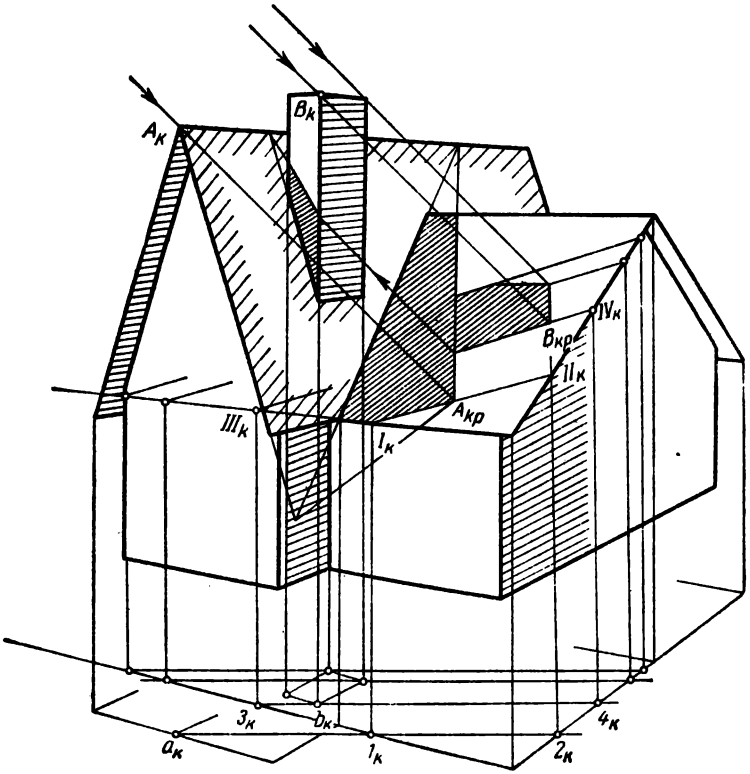

Fig. 501

Example 1. Fig. 503 gives a frontal dimetric view of a cylindrical surface and two vertical planes tangent to it. In front and behind, the surface and planes are bounded by lines I_K and II_K respectively.

Begin defining the shadow by constructing the shadows of lines I_K and II_K.

The shadow of each of them will be the shadows of two lines tangent to the shadow of a semicircle. For example, the shadow of line I_K consists of shadows $a_K A_{KH}$, $b_K B_{KH}$ of two vertical lines and the shadow $A_{KH} C_{KH} B_{KH}$ of the semicircle.

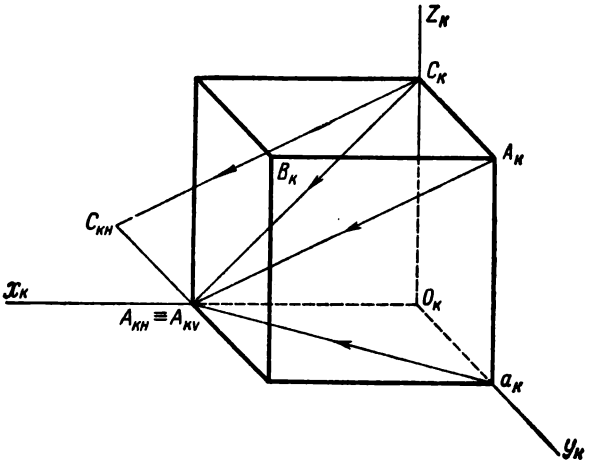

Fig. 502

The line $M_{KH} N_{KH}$ jointly tangent to the curvilinear portions of lines I_{KH} and II_{KH} enables us (by means of reverse rays) to define the limit of shade: the generatrix $M_K N_K$.

This line of shade may also be constructed without resorting to reverse rays.

Indeed, the required line (generatrix $M_K N_K$) is the line along which the radial plane touches the given cylindrical surface.

As we know, the solution of the problem on passing a plane tangent to a cylinder and parallel to a given straight line (in our case parallel to the light ray) begins with the construction of an auxiliary plane **director**. The plane director must be parallel to the generatrices of the cylinder and to the given straight line.

In Fig. 503 this plane (cross-hatched) is defined by a diagonal of the cube (a ray) and the horizontal edge that is parallel to the generatrices of the cylindrical surface. The cross-hatched plane and vertical

plane intersect along a straight line that makes an angle of 45° with the horizon (a diagonal of the rear face of the cube).

This angle determines the direction of the vertical lines of the radial tangent plane P_K.

In the same figure, the vertical line $M_K M_{1K}$ is drawn tangent to the front semicircle.

The point of tangency M_K thus found enables us to construct the desired generatrix $M_K N_K$.

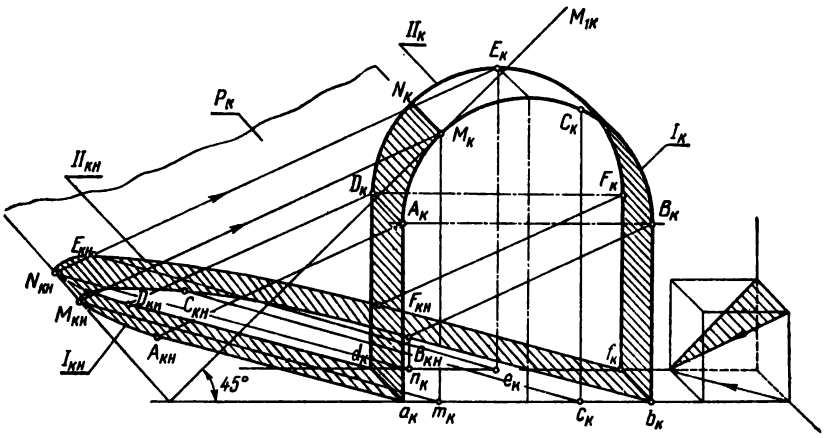

Fig. 503

This construction of the line of shade yields a more accurate result than the method of reverse rays.

Example 2. Fig. 504 shows the construction of shadows of a group of geometric solids. Of the visible faces, those to the right are in shade.

The shadow of the vertical edge $A_K a_K$ on plane $x_K O_K y_K$ will be parallel to the secondary projection of the ray on this plane. This shadow is then deflected and moves vertically upwards along plane U_K to point B_K. In the construction of the shadows $A_{K\Phi}$, $C_{K\Phi}$, $D_{K\Phi}$ and $E_{K\Phi}$ of points on the cylindrical surface we employ the secondary projections of the light ray on plane $y_K O_K z_K$. The shadow of the elliptical arc $F_K L_K N_K$ is the locus of points in which the light rays passing through the points of the arc intersect the coordinate plane $x_K O_K y_K$.

The rectilinear line of shadow of the cylinder is the shade of the generatrix along which the cylinder and radial plane touch. In Fig. 504 the position of this generatrix is defined by point N_K, the point of tangency of the right base of the cylinder and a straight line drawn parallel to the secondary projection of the ray on plane $y_K O_K z_K$.

Example 3. Fig. 505 shows a building outlined in oblique isometry (the picture plane K is parallel to H). The direction of the light ray is determined by the straight line $C_K C_{KH}$, which passes through the centre of the upper right semicircle and, intersecting its secondary projection, yields the shadow C_{KH} of the centre.

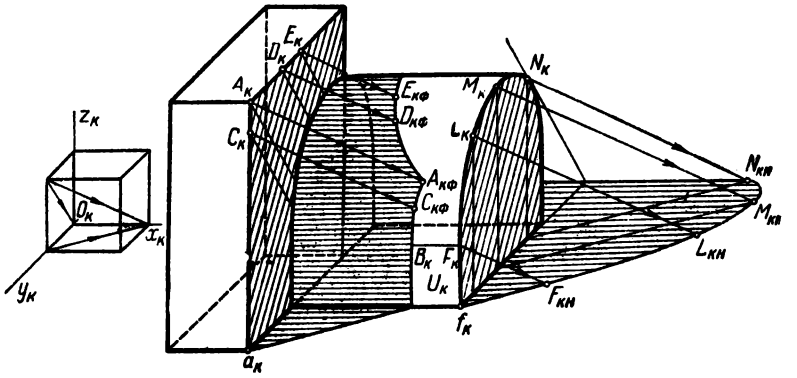

Fig. 504

If the plane of the roof is parallel to H, the semicircle and its shadow on H will be congruent.

The generatrix $E_K e_K$, which separates the illuminated portion of the surface of the right semicylinder from the shaded portion, is the line of tangency of the cylindrical surface and the radial plane. The

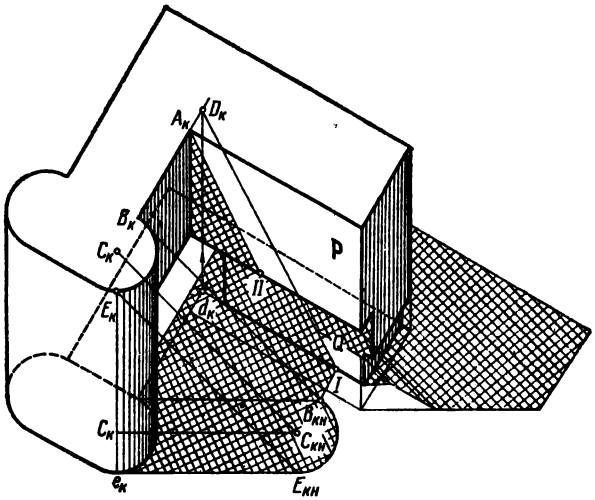

Fig. 505

position of this generatrix is defined by point e_K: the point of tangency of the lower base of the cylinder and the horizontal trace $e_K E_{KH}$ of the radial tangent plane.

In our case, when the generatrices of the cylinder are perpendicular to plane H, the horizontal trace of the radial plane will be parallel to the secondary projection of the light ray, that is, $e_K E_{KH}$ is parallel to $c_K C_{KH}$.

Draw the shadow of edge $A_K B_K$ in the H plane parallel to the edge itself (B_{KH}-I is parallel to $A_K B_K$) to wall Q, where the shadow is deflected in point I and moves to D_K.

Point D_K is the point of intersection of line $A_K B_K$ and the vertical wall Q. In Fig. 505 we first determined the secondary projection d_K of this point (the intersection of the secondary projection of the straight line and the trace of plane Q). The shadow of this same edge $A_K B_K$ on plane P will pass through point A_K parallel to the straight line D_K-I; in other words, A_K-II is parallel to D_K-I.

The foregoing examples do not exhaust by far the possible cases encountered in shade and shadow construction. This problem is dealt with in more detail in the special literature.

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